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Effective gravitational couplings for cosmological perturbations in the most general scalar–tensor theories with second-order field equations

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ABSTRACT

In the Horndeski's most general scalar–tensor theories the equations of scalar density perturbations are derived in the presence of non-relativistic matter minimally coupled to gravity. Under a quasi-static approximation on sub-horizon scales we obtain the effective gravitational coupling G_{eff} associated with the growth rate of matter perturbations as well as the effective gravitational potential Φ_{eff} relevant to the deviation of light rays. We then apply our formulas to a number of modified gravitational models of dark energy – such as those based on $f(R)$ theories, Brans–Dicke theories, kinetic gravity braidings, covariant Galileons, and field derivative couplings with the Einstein tensor. Our results are useful to test the large-distance modification of gravity from the future high-precision observations of large-scale structure, weak lensing, and cosmic microwave background.

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1. Introduction

The late-time cosmic acceleration has been supported by several independent observations – such as supernovae Ia [1], Cosmic Microwave Background (CMB) [2], and baryon acoustic oscillations [3]. The simplest candidate for dark energy is the cosmological constant, but the typical scale of the vacuum energy is vastly larger than the observed energy scale of dark energy [4]. Instead many alternative models have been proposed to identify the origin of dark energy [5,6].

A minimally coupled scalar field with a potential $V(\phi)$ – quintessence [7] – can account for the cosmic acceleration today, provided that the potential is sufficiently flat with a small effective mass $m_\phi \approx 10^{-33}$ eV. The k-essence [8], where the Lagrangian includes a nonlinear term of the field kinetic energy, can be responsible for dark energy even in the absence of a field potential. Quintessence and k-essence can be distinguished from the cosmological constant in that their equations of state vary in time while the latter does not.

There is another class of dark energy models based on the large-distance modification of gravity – such as (i) $f(R)$ theories [9], (ii) Brans–Dicke theories [10], (iii) Dvali–Gabadadze–Porrati (DGP) braneworld [11], and (iv) Galileon gravity [12]. In the local region where the average density is much larger than the cosmological one, these models need to recover the General Relativistic behavior for consistency with solar-system experiments [13]. The models based on the theories (i) and (ii) can be made to be compatible with local gravity constraints under the chameleon mechanism [14], as long as the scalar degree of freedom has a large effective mass in the region of high density [15]. For the models based on the theories (iii) and (iv) the nonlinear field self-interaction can allow the recovery of the General Relativistic behavior in the local region [16] through the Vainshtein mechanism [17].

For the dark energy models mentioned above the field equations of motion are kept up to second order. This is desirable to avoid the appearance of the Ostrogradski's instability [18] associated with the derivatives higher than the second order. In 1974 Horndeski [19] derived the most general single-field Lagrangian for scalar–tensor theories with second-order equations of motion. Recently this issue was revisited by Deffayet et al. [20] in connection to a covariant Galileon field. The most general scalar–tensor theories with the second-order equations can be expressed by the sum of the Lagrangians (2)–(5) below. In fact one can show that this Lagrangian is equivalent to that derived by Horndeski [21] (see also Ref. [22]).

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The most general scalar–tensor theories not only include quintessence and k-essence but also accommodate $f(R)$ theories, Brans–Dicke theories, and Galileon gravity. Moreover, as shown in Ref. [21], several different choices of the functions K , G_i ($i = 3, 4, 5$) give rise to the (modified) DGP model in 4 dimensions [23], the field coupling with the Gauss–Bonnet term [24], the field derivative coupling with the Einstein tensor [25,26], and so on.

In this Letter we shall derive the equations of linear density perturbations for the most general scalar–tensor theories with non-relativistic matter taken into account. In the presence of the terms \mathcal{L}_i ($i = 3, 4, 5$) the effective gravitational coupling G_{eff} is subject to change compared to that in General Relativity. This leads to the modified growth rate of matter density perturbations δ_m as well as the modified evolution of the effective gravitational potential Φ_{eff} associated with the deviation of light rays. Similar analysis has been carried out in specific scalar–tensor theories [6,27–31], $f(R)$ theories [31,32], kinetic gravity braidings with the term \mathcal{L}_3 [33–37], and covariant Galileon [38]. Our analysis in this Letter covers those theories as specific cases. Such general analysis will be useful to discriminate between modified gravitational models from the observations of large-scale structure, weak lensing, and CMB [39].

This Letter is organized as follows. In Section 2 we derive the background equations of motion on the flat Friedmann–Lemaître–Robertson–Walker (FLRW) background for the action (6) below. In Section 3 we obtain the full scalar perturbation equations of motion for the metric (21). In Section 4 the effective gravitational coupling G_{eff} as well as the effective gravitational potential Φ_{eff} are derived under the quasi-static approximation on sub-horizon scales. In Section 5 we apply our formulas of sub-horizon perturbations to a number of modified gravitational models of dark energy. Section 6 is devoted to conclusions.

2. The most general scalar–tensor theories and the background equations of motion

The most general 4-dimensional scalar–tensor theories keeping the field equations of motion at second order are described by the Lagrangian [20]

$$\mathcal{L} = \sum_{i=2}^5 \mathcal{L}_i, \quad (1)$$

where

$$\mathcal{L}_2 = K(\phi, X), \quad (2)$$

$$\mathcal{L}_3 = -G_3(\phi, X)\square\phi, \quad (3)$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4,X}[(\square\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)], \quad (4)$$

$$\mathcal{L}_5 = G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}G_{5,X}[(\square\phi)^3 - 3(\square\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)]. \quad (5)$$

Here K and G_i ($i = 3, 4, 5$) are functions in terms of a scalar field ϕ and its kinetic energy $X = -\partial^\mu\phi\partial_\mu\phi/2$ with the partial derivatives $G_{i,X} \equiv \partial G_i/\partial X$, R is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor. The above Lagrangian was first discovered by Horndeski in a different form [19]. In fact the Lagrangian (1) is equivalent to that derived by Horndeski [21].

We are interested in the late-time cosmology in which the field ϕ is responsible for dark energy. In addition we take into account a barotropic perfect fluid with the equation of state $w = P_m/\rho_m$, where P_m is the pressure and ρ_m is the energy density respectively. In the following we focus on non-relativistic matter ($w = 0$) minimally coupled to the field ϕ .¹ The total action we are going to study is then given by

$$S = \int d^4x \sqrt{-g}(\mathcal{L} + \mathcal{L}_m), \quad (6)$$

where g is a determinant of the metric $g_{\mu\nu}$, and \mathcal{L}_m is the Lagrangian of non-relativistic matter.

Let us consider a flat FLRW background with the metric $ds^2 = -N^2(t)dt^2 + a^2(t)d\mathbf{x}^2$. Variations with respect to the lapse $N(t)$ and the scale factor $a(t)$ give rise to the following equations of motion respectively

$$\mathcal{E} \equiv \sum_{i=2}^5 \mathcal{E}_i = -\rho_m, \quad (7)$$

$$\mathcal{P} \equiv \sum_{i=2}^5 \mathcal{P}_i = 0, \quad (8)$$

where

$$\mathcal{E}_2 \equiv 2XK_{,X} - K, \quad (9)$$

$$\mathcal{E}_3 \equiv 6X\dot{\phi}HG_{3,X} - 2XG_{3,\phi}, \quad (10)$$

$$\mathcal{E}_4 \equiv -6H^2G_4 + 24H^2X(G_{4,X} + XG_{4,XX}) - 12HX\dot{\phi}G_{4,\phi X} - 6H\dot{\phi}G_{4,\phi}, \quad (11)$$

$$\mathcal{E}_5 \equiv 2H^3X\dot{\phi}(5G_{5,X} + 2XG_{5,XX}) - 6H^2X(3G_{5,\phi} + 2XG_{5,\phi X}), \quad (12)$$

¹ If matter is non-minimally coupled to ϕ through the coupling to the metric $\tilde{g}_{\mu\nu} = A(\phi)g_{\mu\nu}$ rather than $g_{\mu\nu}$, one rewrites the action (6) in terms of ϕ and $\tilde{g}_{\mu\nu}$ rather than ϕ and $g_{\mu\nu}$. Then, the Lagrangian is still of the form (2)–(5), because the change of the variable $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = A(\phi)g_{\mu\nu}$ does not generate higher derivative terms in the field equations. For this reason, it is sufficient to consider matter minimally coupled to ϕ .

and

$$\mathcal{P}_2 \equiv K, \quad (13)$$

$$\mathcal{P}_3 \equiv -2X(G_{3,\phi} + \ddot{\phi}G_{3,X}), \quad (14)$$

$$\begin{aligned} \mathcal{P}_4 \equiv & 2(3H^2 + 2\dot{H})G_4 - 12H^2XG_{4,X} - 4H\dot{X}G_{4,X} - 8\dot{H}XG_{4,X} - 8HX\dot{X}G_{4,XX} + 2(\ddot{\phi} + 2H\dot{\phi})G_{4,\phi} \\ & + 4XG_{4,\phi\phi} + 4X(\ddot{\phi} - 2H\dot{\phi})G_{4,\phi X}, \end{aligned} \quad (15)$$

$$\begin{aligned} \mathcal{P}_5 \equiv & -2X(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 3H^2\ddot{\phi})G_{5,X} - 4H^2X^2\ddot{\phi}G_{5,XX} + 4HX(\dot{X} - HX)G_{5,\phi X} \\ & + 2[2(\dot{H}X + H\dot{X}) + 3H^2X]G_{5,\phi} + 4HX\dot{\phi}G_{5,\phi\phi}. \end{aligned} \quad (16)$$

Here a dot represents a derivative with respect to t and $H \equiv \dot{a}/a$ is the Hubble parameter. Varying the action (6) with respect to $\phi(t)$, it follows that

$$\frac{1}{a^3} \frac{d}{dt} (a^3 J) = P_\phi, \quad (17)$$

where

$$\begin{aligned} J \equiv & \dot{\phi}K_{,X} + 6HXG_{3,X} - 2\dot{\phi}G_{3,\phi} + 6H^2\dot{\phi}(G_{4,X} + 2XG_{4,XX}) - 12HXG_{4,\phi X} \\ & + 2H^3X(3G_{5,X} + 2XG_{5,XX}) - 6H^2\dot{\phi}(G_{5,\phi} + XG_{5,\phi X}), \end{aligned} \quad (18)$$

$$\begin{aligned} P_\phi \equiv & K_{,\phi} - 2X(G_{3,\phi\phi} + \ddot{\phi}G_{3,\phi X}) + 6(2H^2 + \dot{H})G_{4,\phi} + 6H(\dot{X} + 2HX)G_{4,\phi X} \\ & - 6H^2XG_{5,\phi\phi} + 2H^3X\dot{\phi}G_{5,\phi X}. \end{aligned} \quad (19)$$

Non-relativistic matter obeys the continuity equation

$$\dot{\rho}_m + 3H\rho_m = 0. \quad (20)$$

Eqs. (7), (8), (17), and (20) are not independent because of the Bianchi identities. In fact the field equation (17) can be derived by using Eqs. (7), (8), and (20).

3. Linear perturbation equations

In this section we derive the linear perturbation equations for the theories give by the action (6). Let us consider the following perturbed metric about the flat FLRW background [40]

$$ds^2 = -(1 + 2\psi) dt^2 - 2\partial_i \chi dt dx^i + a^2(t)(1 + 2\Phi)\delta_{ij} dx^i dx^j, \quad (21)$$

where ψ , Φ , and χ are the scalar metric perturbations. In this expression we have chosen a spatial gauge such that g_{ij} is diagonal, which fixes the spatial part of a vector associated with a scalar gauge transformation. The temporal part of the gauge-transformation vector is not fixed for the moment.

We perturb the scalar field as $\phi(t) + \delta\phi(t, \mathbf{x})$, and the matter fields as well, in terms of the matter density perturbation $\delta\rho_m$, and the scalar part of the fluid velocity v . We define the density contrast of matter as $\delta \equiv \delta\rho_m/\rho_m$. In order to write the perturbation equations in a compact form, we introduce the following quantities [21],

$$\mathcal{F}_T \equiv 2[G_4 - X(\ddot{\phi}G_{5,X} + G_{5,\phi})], \quad (22)$$

$$\mathcal{G}_T \equiv 2[G_4 - 2XG_{4,X} - X(H\dot{\phi}G_{5,X} - G_{5,\phi})], \quad (23)$$

and

$$\begin{aligned} \Theta \equiv & -\frac{1}{6} \frac{\partial \mathcal{E}}{\partial H} \\ = & -\dot{\phi}XG_{3,X} + 2HG_4 - 8HXG_{4,X} - 8HX^2G_{4,XX} + \dot{\phi}G_{4,\phi} + 2X\dot{\phi}G_{4,\phi X} \\ & - H^2\dot{\phi}(5XG_{5,X} + 2X^2G_{5,XX}) + 2HX(3G_{5,\phi} + 2XG_{5,\phi X}), \end{aligned} \quad (24)$$

$$\begin{aligned} \Sigma \equiv & X \frac{\partial \mathcal{E}}{\partial X} + \frac{1}{2} H \frac{\partial \mathcal{E}}{\partial H} \\ = & XK_{,X} + 2X^2K_{,XX} + 12H\dot{\phi}XG_{3,X} + 6H\dot{\phi}X^2G_{3,XX} - 2XG_{3,\phi} - 2X^2G_{3,\phi X} \\ & - 6H^2G_4 + 6[H^2(7XG_{4,X} + 16X^2G_{4,XX} + 4X^3G_{4,XXX}) - H\dot{\phi}(G_{4,\phi} + 5XG_{4,\phi X} + 2X^2G_{4,\phi XX})] \\ & + 30H^3\dot{\phi}XG_{5,X} + 26H^3\dot{\phi}X^2G_{5,XX} + 4H^3\dot{\phi}X^3G_{5,XXX} - 6H^2X(6G_{5,\phi} + 9XG_{5,\phi X} + 2X^2G_{5,\phi XX}), \end{aligned} \quad (25)$$

where we used the relation $X\partial_X\dot{\phi} = \dot{\phi}/2$. The functions \mathcal{F}_T and \mathcal{G}_T appear in the quadratic action for the cosmological tensor perturbations [21,41,42]. In order to avoid ghost and Laplacian instabilities in the tensor sector we require the conditions $\mathcal{F}_T > 0$ and $\mathcal{G}_T > 0$.

We expand the action (6) up to second order in perturbations and vary the second-order action with respect to each perturbed variable such as ψ . Following the procedure explained in Ref. [38], the perturbation equations in Fourier space are given by

$$E_\Psi \equiv A_1 \dot{\Phi} + A_2 \delta\dot{\Phi} - \rho_m \dot{\nu} + A_3 \frac{k^2}{a^2} \Phi + A_4 \Psi + A_5 \frac{k^2}{a^2} \chi + \left(A_6 \frac{k^2}{a^2} - \mu \right) \delta\phi - \rho_m \delta = 0, \quad (26)$$

$$E_\phi \equiv B_1 \ddot{\Phi} + B_2 \delta\ddot{\Phi} + B_3 \dot{\Phi} + B_4 \delta\dot{\Phi} + B_5 \dot{\Psi} + B_6 \frac{k^2}{a^2} \Phi + \left(B_7 \frac{k^2}{a^2} + 3\nu \right) \delta\phi + \left(B_8 \frac{k^2}{a^2} + B_9 \right) \Psi + B_{10} \frac{k^2}{a^2} \dot{\chi} + B_{11} \frac{k^2}{a^2} \chi + 3\rho_m \dot{\nu} = 0, \quad (27)$$

$$E_\chi \equiv C_1 \dot{\Phi} + C_2 \delta\dot{\Phi} + C_3 \Psi + C_4 \delta\phi + \rho_m \nu = 0, \quad (28)$$

$$E_{\delta\phi} \equiv D_1 \ddot{\Phi} + D_2 \delta\ddot{\Phi} + D_3 \dot{\Phi} + D_4 \delta\dot{\Phi} + D_5 \dot{\Psi} + D_6 \frac{k^2}{a^2} \dot{\chi} + \left(D_7 \frac{k^2}{a^2} + D_8 \right) \Phi + \left(D_9 \frac{k^2}{a^2} - M^2 \right) \delta\phi + \left(D_{10} \frac{k^2}{a^2} + D_{11} \right) \Psi + D_{12} \frac{k^2}{a^2} \chi = 0, \quad (29)$$

$$E_\nu \equiv \dot{\nu} - \Psi = 0, \quad (30)$$

$$E_\delta \equiv \dot{\delta} + 3\dot{\Phi} + \frac{k^2}{a^2} \nu - \frac{k^2}{a^2} \chi = 0, \quad (31)$$

where k is a comoving wavenumber, and

$$A_1 = 6\Theta, \quad A_2 = -2(\Sigma + 3H\Theta)/\dot{\phi}, \quad A_3 = 2\mathcal{G}_T, \quad A_4 = 2\Sigma + \rho_m, \quad A_5 = -2\Theta, \quad (32)$$

$$A_6 = 2(\Theta - H\mathcal{G}_T)/\dot{\phi}, \quad \mu = \mathcal{E}_{,\phi},$$

$$B_1 = 6\mathcal{G}_T, \quad B_2 = 6(\Theta - H\mathcal{G}_T)/\dot{\phi}, \quad B_3 = 6(\dot{\mathcal{G}}_T + 3H\mathcal{G}_T),$$

$$B_4 = 3[(4H\ddot{\phi} - 4\dot{H}\dot{\phi} - 6H^2\dot{\phi})\mathcal{G}_T - 2H\dot{\phi}\dot{\mathcal{G}}_T - (4\ddot{\phi} - 6H\dot{\phi})\Theta + 2\dot{\phi}\dot{\Theta} - \rho_m\dot{\phi}]/\dot{\phi}^2, \quad B_5 = -6\Theta,$$

$$B_6 = 2\mathcal{F}_T, \quad B_7 = 2[\dot{\mathcal{G}}_T + H(\mathcal{G}_T - \mathcal{F}_T)]/\dot{\phi}, \quad B_8 = 2\mathcal{G}_T, \quad B_9 = -6(\dot{\Theta} + 3H\Theta),$$

$$B_{10} = -2\mathcal{G}_T, \quad B_{11} = -2(\dot{\mathcal{G}}_T + H\mathcal{G}_T), \quad \nu = \mathcal{P}_{,\phi}, \quad (33)$$

$$C_1 = 2\mathcal{G}_T, \quad C_2 = 2(\Theta - H\mathcal{G}_T)/\dot{\phi}, \quad C_3 = -2\Theta, \quad C_4 = [2(H\ddot{\phi} - \dot{H}\dot{\phi})\mathcal{G}_T - 2\ddot{\phi}\Theta - \rho_m\dot{\phi}]/\dot{\phi}^2, \quad (34)$$

$$D_1 = 6(\Theta - H\mathcal{G}_T)/\dot{\phi}, \quad D_2 = 2(3H^2\mathcal{G}_T - 6H\Theta - \Sigma)/\dot{\phi}^2,$$

$$D_3 = -3[2H(\dot{\mathcal{G}}_T + 3H\mathcal{G}_T) - 2(\dot{\Theta} + 3H\Theta) - \rho_m]/\dot{\phi},$$

$$D_4 = 2[3H\{(3H^2 + 2\dot{H})\dot{\phi} - 2H\ddot{\phi}\}\mathcal{G}_T + 3H^2\dot{\phi}\dot{\mathcal{G}}_T + 6\{2H\ddot{\phi} - (3H^2 + \dot{H})\dot{\phi}\}\Theta - 6H\dot{\phi}\dot{\Theta} + (2\ddot{\phi} - 3H\dot{\phi})\Sigma - \dot{\phi}\dot{\Sigma}]/\dot{\phi}^3,$$

$$D_5 = 2(\Sigma + 3H\Theta)/\dot{\phi}, \quad D_6 = -2(\Theta - H\mathcal{G}_T)/\dot{\phi}, \quad D_7 = 2[\dot{\mathcal{G}}_T + H(\mathcal{G}_T - \mathcal{F}_T)]/\dot{\phi},$$

$$D_8 = 3[6(\dot{H}\dot{\phi} - H\ddot{\phi})\Theta - 2\ddot{\phi}\Sigma + 3H\rho_m\dot{\phi} - \mu\dot{\phi}^2]/\dot{\phi}^2,$$

$$D_9 = [2H^2\mathcal{F}_T - 4H(\dot{\mathcal{G}}_T + H\mathcal{G}_T) + 2(\dot{\Theta} + H\Theta) + \rho_m]/\dot{\phi}^2,$$

$$D_{10} = 2(\Theta - H\mathcal{G}_T)/\dot{\phi}, \quad D_{11} = [6\{(3H^2 + \dot{H})\dot{\phi} - H\ddot{\phi}\}\Theta + 6H\dot{\phi}\dot{\Theta} + 2(3H\dot{\phi} - \ddot{\phi})\Sigma + 2\dot{\phi}\dot{\Sigma} - \mu\dot{\phi}^2]/\dot{\phi}^2,$$

$$D_{12} = [2H(\dot{\mathcal{G}}_T + H\mathcal{G}_T) - 2(\dot{\Theta} + H\Theta) - \rho_m]/\dot{\phi},$$

$$M^2 = [\dot{\mu} + 3H(\mu + \nu)]/\dot{\phi}$$

$$\begin{aligned} &= -K_{,\phi\phi} + (\ddot{\phi} + 3H\dot{\phi})K_{,\phi\chi} + 2XK_{,\phi\phi\chi} + 2X\ddot{\phi}K_{,\phi\chi\chi} \\ &+ [6H(G_{3,\phi\chi\chi}X + G_{3,\phi\chi})\dot{\phi} - 2G_{3,\phi\phi\chi}X - 2G_{3,\phi\phi}] \ddot{\phi} + 6H(G_{3,\phi\phi\chi}X - G_{3,\phi\phi})\dot{\phi} \\ &+ 6G_{3,\phi\chi}X\dot{H} + 2(9H^2G_{3,\phi\chi} - G_{3,\phi\phi\phi})X \\ &+ [6H^2(4G_{4,\phi\chi\chi}X^2 + 8G_{4,\phi\chi\chi}X + G_{4,\phi\chi}) - 6H(2G_{4,\phi\phi\chi\chi}X + 3G_{4,\phi\phi\chi})\dot{\phi}] \ddot{\phi} \\ &+ [12H(G_{4,\phi\chi} + 2G_{4,\phi\chi\chi})\dot{H} + 6H(6H^2G_{4,\phi\chi\chi}X - 2G_{4,\phi\phi\phi\chi}X + 3H^2G_{4,\phi\chi})] \dot{\phi} \\ &+ 12H^2(2G_{4,\phi\phi\chi\chi}X^2 - 3G_{4,\phi\phi\chi}X - G_{4,\phi\phi}) - 6(2G_{4,\phi\phi\chi}X + G_{4,\phi\phi})\dot{H} \\ &+ [2H^3(2G_{5,\phi\chi\chi}X^2 + 7G_{5,\phi\chi\chi}X + 3G_{5,\phi\chi})\dot{\phi} - 6H^2(5G_{5,\phi\phi\chi}X + G_{5,\phi\phi} + 2G_{5,\phi\phi\chi\chi}X^2)] \ddot{\phi} \\ &+ [2H^3(2G_{5,\phi\phi\chi\chi}X^2 - 9G_{5,\phi\phi} - 7G_{5,\phi\phi\chi}X) - 12H(G_{5,\phi\phi\chi}X + G_{5,\phi\phi})\dot{H}] \dot{\phi} \\ &+ 6H^2X(3G_{5,\phi\chi} + 2G_{5,\phi\chi\chi})\dot{H} + 6H^2X(3H^2G_{5,\phi\chi} - G_{5,\phi\phi\phi} + 2H^2G_{5,\phi\chi\chi}X - 2G_{5,\phi\phi\phi}X). \end{aligned} \quad (35)$$

In deriving the above we used the background equations. The expression of the coefficients of Eqs. (26)–(29), written in terms of the variables Θ , Σ , etc., becomes compact, though many of the coefficients include terms $\dot{\phi}^n$ ($n > 0$) in the denominators. However, there are no divergences at $\dot{\phi} = 0$. In fact, whenever the term $\dot{\phi}^n$ appears in the denominator, the numerator of the same coefficient compensates with the term $\dot{\phi}^n$. For instance, this property can be seen in the expression of M^2 . From the expressions of the coefficients given above, it is clear that not all these coefficients are independent. For example, later on, we will find it convenient to use the following relations $A_3 = B_8$, $D_7 = B_7$, and $D_{10} = A_6$.

The mass of the field $\delta\phi$ is related with the term M^2 defined in Eq. (35). In fact, for a canonical scalar field described by the Lagrangian $K = X - V(\phi)$ with $G_i = 0$ ($i = 3, 4, 5$), we have that $M^2 = -K_{,\phi\phi} = V_{,\phi\phi}$. In viable dark energy models based on $f(R)$ gravity and Brans–Dicke theory with a field potential, the term $-K_{,\phi\phi}$ is the dominant contribution to M^2 [43,44]. The term $-K_{,\phi\phi}$ comes from the time-derivative of μ , such that the contribution from the term $3H(\mu + \nu)$ is usually unimportant relative to $\dot{\mu}$.

The equations of motion (26)–(31), are not independent. In fact, we find the identity

$$\dot{E}_\Psi + 3HE_\Psi - HE_\Phi - \frac{k^2}{a^2}E_\chi - \dot{\phi}E_{\delta\phi} + \rho_m(\dot{E}_\nu + 3HE_\nu) + \rho_m E_\delta = 0. \quad (36)$$

This relation can be used in two ways: 1) to check the consistency of the equations themselves; 2) to get some equations of motion, which would be missing when some gauge is used from the beginning. For example, in the Newtonian gauge ($\chi = 0$), the equation $E_\chi|_{\chi=0} = 0$, cannot be derived directly. However, it is still possible to obtain it by using Eq. (36).

We note that the following combination of the perturbation equations is useful:

$$\frac{k^2}{a^2}\tilde{E}_\gamma \equiv 3(\dot{E}_\chi + 3HE_\chi) - E_\Phi = 0, \quad (37)$$

which is written explicitly as

$$\tilde{E}_\gamma = B_6\Phi + B_7\delta\phi + B_8\Psi + B_{10}\dot{\chi} + B_{11}\chi = 0. \quad (38)$$

This equation corresponds to the traceless part of the gravitational field equations.

In order to study the evolution of matter perturbations we introduce the gauge-invariant density contrast

$$\delta_m \equiv \delta + 3H\nu. \quad (39)$$

From Eqs. (30) and (31) it follows that

$$\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{k^2}{a^2}(\Psi - \dot{\chi}) = 3(\ddot{I} + 2H\dot{I}), \quad (40)$$

where $I \equiv H\nu - \Phi$.

4. Effective gravitational couplings under the quasi-static approximation on sub-horizon scales

When we discuss the evolution of matter perturbations relevant to large-scale structure and weak lensing, we are primarily interested in the modes deep inside the Hubble radius ($k^2/a^2 \gg H^2$). We shall use the quasi-static approximation on sub-horizon scales,² under which the dominant contributions in the perturbation equations correspond to those including k^2/a^2 and δ [6,27,45]. There are two different classes of dark energy models, depending on the mass M of a scalar degree of freedom.

The first one corresponds to the case in which the mass M becomes large in the early cosmological epoch. The viable dark energy models constructed in the framework of $f(R)$ gravity [43,47,48] and Brans–Dicke theories [44] belong to this class. Since the effect of the field mass cannot be neglected in such cases, we need to take into account the term M^2 to discuss the evolution of perturbations. This induces the oscillation of the field perturbation $\delta\phi$, but as long as this oscillating mode is initially suppressed relative to the matter-induced mode, the quasi-static approximation can reproduce numerically integrated solutions with high accuracy [43,49]. Under the quasi-static approximation we neglect the time-derivatives of $\delta\phi$, which corresponds to the approximation under which the oscillating mode is unimportant relative to the matter-induced mode.

Another class corresponds to the case in which the field does not have a massive potential, e.g., Galileon gravity. In such cases the numerical simulations in Refs. [35,38] also show that the quasi-static approximation is sufficiently accurate for the modes deep inside the Hubble radius.

We expect that the quasi-static approximation on sub-horizon scales should be trustable for our general theories as well, provided that the matter-induced mode dominates over the oscillating mode.

Let us choose the Newtonian gauge in which $\chi = 0$. Under the quasi-static approximation on sub-horizon scales we find that Eqs. (38), (29), and (26) can be rewritten as

$$B_6\Phi + B_7\delta\phi + B_8\Psi = 0, \quad (41)$$

$$B_7\frac{k^2}{a^2}\Phi + \left(D_9\frac{k^2}{a^2} - M^2\right)\delta\phi + A_6\frac{k^2}{a^2}\Psi \simeq 0, \quad (42)$$

$$B_8\frac{k^2}{a^2}\Phi + A_6\frac{k^2}{a^2}\delta\phi - \rho_m\delta \simeq 0, \quad (43)$$

where we used the relations $A_3 = B_8$, $D_7 = B_7$, and $D_{10} = A_6$ already mentioned in Section 3. The above three equations correspond to the traceless part of the gravitational field equations, the scalar-field equation of motion, and the (00)-component of the gravitational field equations, respectively. Note that Eq. (41) also follows from Eq. (27) in the same approximation scheme.

² Strictly speaking, the typical scale here should be given by the sound horizon rather than the Hubble horizon because the propagation speed of the scalar mode, c_s , differs from unity in general [21,46]. One needs to be careful for the use of the quasi-static approximation in models with $c_s \ll 1$, as the range of the validity of the approximation may be quite limited.

We can solve Eqs. (41) and (42) for Φ and $\delta\phi$ in terms of Ψ , and then substitute these expressions into Eq. (43). This gives the following Poisson equation

$$\frac{k^2}{a^2}\Psi \simeq -4\pi G_{\text{eff}}\rho_m\delta. \quad (44)$$

Here the effective gravitational coupling G_{eff} is given by

$$G_{\text{eff}} = \frac{2M_{\text{pl}}^2[(B_6D_9 - B_7^2)(k/a)^2 - B_6M^2]}{(A_6^2B_6 + B_8^2D_9 - 2A_6B_7B_8)(k/a)^2 - B_8^2M^2} G \quad (45)$$

$$= \frac{M_{\text{pl}}^2\{(\dot{\Theta} + H\Theta)\mathcal{F}_S + (\dot{\mathcal{G}}_T - \dot{\Theta}\mathcal{G}_T/\Theta)^2 + \mathcal{F}_T[XM^2a^2/k^2 + (\mathcal{E} + \mathcal{P})/2]\}}{\Theta^2\mathcal{F}_S + \mathcal{G}_T^2[XM^2a^2/k^2 + (\mathcal{E} + \mathcal{P})/2]} G, \quad (46)$$

where G is the bare gravitational constant related with the reduced Planck mass M_{pl} via the relation $8\pi G = M_{\text{pl}}^{-2}$, and

$$\mathcal{F}_S \equiv \frac{1}{a} \frac{d}{dt} \left(\frac{a}{\Theta} \mathcal{G}_T^2 \right) - \mathcal{F}_T. \quad (47)$$

In order to avoid the Laplacian instability of scalar perturbations we require that $\mathcal{F}_S > 0$ [21,41,42]. While G_{eff} is written in a compact expression in Eq. (46), it is often convenient to use the form (45) for a given Lagrangian. In Appendix A we present the explicit forms of the coefficients A_6 , B_6 , B_7 , B_8 , and D_9 , which is useful for the computation of Eq. (45).

Under the quasi-static approximation on sub-horizon scales Eq. (40) gives

$$\ddot{\delta}_m + 2H\dot{\delta}_m + \frac{k^2}{a^2}\Psi \simeq 0. \quad (48)$$

On using Eq. (44) and $\delta_m \simeq \delta$ (which are valid for $k^2/a^2 \gg H^2$), it follows that

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\text{eff}}\rho_m\delta_m \simeq 0. \quad (49)$$

This can be written as

$$\delta_m'' + \left(2 + \frac{H'}{H}\right)\delta_m' - \frac{3}{2} \frac{G_{\text{eff}}}{G} \Omega_m \delta_m \simeq 0, \quad (50)$$

where $\Omega_m \equiv \rho_m/(3M_{\text{pl}}^2H^2)$, and a prime represents a derivative with respect to $N = \ln a$.

We define the anisotropic parameter η to characterize the difference between the two gravitational potentials:

$$\eta \equiv -\Phi/\Psi. \quad (51)$$

Under the quasi-static approximation on sub-horizon scales this reduces to

$$\eta \simeq \frac{(B_8D_9 - A_6B_7)(k/a)^2 - B_8M^2}{(B_6D_9 - B_7^2)(k/a)^2 - B_6M^2} \quad (52)$$

$$= \frac{\mathcal{G}_T\dot{\Theta} + (H\mathcal{F}_T - \dot{\mathcal{G}}_T)\Theta - H\mathcal{G}_T(\dot{\mathcal{G}}_T + H\mathcal{G}_T) - \mathcal{G}_T[(\mathcal{E} + \mathcal{P})/2 + XM^2a^2/k^2]}{\mathcal{F}_T(\dot{\Theta} + H\Theta) - (\dot{\mathcal{G}}_T + H\mathcal{G}_T)^2 - \mathcal{F}_T[(\mathcal{E} + \mathcal{P})/2 + XM^2a^2/k^2]}. \quad (53)$$

We also introduce the effective gravitational potential

$$\Phi_{\text{eff}} \equiv (\Psi - \Phi)/2, \quad (54)$$

which is associated with the deviation of the light rays in CMB and weak lensing observations [50]. From Eqs. (44) and (51) we have

$$\Phi_{\text{eff}} \simeq -4\pi G_{\text{eff}} \frac{1 + \eta}{2} \left(\frac{a}{k}\right)^2 \rho_m \delta \simeq -\frac{3}{2} \frac{G_{\text{eff}}}{G} \frac{1 + \eta}{2} \left(\frac{aH}{k}\right)^2 \Omega_m \delta_m. \quad (55)$$

Let us consider k-essence in the framework of General Relativity (GR), which corresponds to the Lagrangian $\mathcal{L} = K(\phi, X) + (M_{\text{pl}}^2/2)R$ [i.e. $G_4 = M_{\text{pl}}^2/2$, $G_3 = 0 = G_5$]. In this case one has $\mathcal{F}_T = \mathcal{G}_T = M_{\text{pl}}^2$, $\Theta = HM_{\text{pl}}^2$, and $\mathcal{F}_S = -M_{\text{pl}}^2\dot{H}/H^2$, which gives $G_{\text{eff}} = G$ and $\eta = 1$ from Eqs. (46) and (53). During the matter-dominated epoch ($H'/H \simeq -3/2$ and $\Omega_m \simeq 1$) there is a growing mode solution $\delta_m \propto a$ to Eq. (50). For this solution $\Phi_{\text{eff}} = \text{constant}$ from Eq. (55). In modified gravitational theories G_{eff} and η are in general different from G and 1 respectively, so that the evolution of δ_m and Φ_{eff} is subject to change compared to GR.

5. Application to specific theories

In this section we apply our formulas of G_{eff} and η derived in Section 4 to a number of modified gravitational theories.

5.1. $f(R)$ theories

The Lagrangian of $f(R)$ theories corresponds to $\mathcal{L} = (M_{\text{pl}}^2/2)f(R)$, where f is an arbitrary function in terms of the Ricci scalar R . This is equivalent to the Lagrangian (1) by choosing the following functions [51]

$$K = -\frac{M_{\text{pl}}^2}{2}(Rf_{,R} - f), \quad G_3 = 0 = G_5, \quad G_4 = \frac{1}{2}M_{\text{pl}}\phi, \quad \phi = M_{\text{pl}}f_{,R}, \quad (56)$$

where ϕ is a scalar degree of freedom having the dimension of mass. Since $G_4 = \phi M_{\text{pl}}/2$, $B_7 = 2A_6 = 2M_{\text{pl}}$, $B_6 = B_8 = 2M_{\text{pl}}\phi$, and $D_9 = 0$ in this case, Eqs. (45) and (52) read

$$G_{\text{eff}} = \frac{M_{\text{pl}}}{\phi} \frac{4 + 2(\phi/M_{\text{pl}})(Ma/k)^2}{3 + 2(\phi/M_{\text{pl}})(Ma/k)^2} G, \quad \eta = \frac{1 + (\phi/M_{\text{pl}})(Ma/k)^2}{2 + (\phi/M_{\text{pl}})(Ma/k)^2}. \quad (57)$$

From Eq. (35) the mass squared of the scalar degree of freedom is given by

$$M^2 = -K_{,\phi\phi} = \frac{1}{2f_{,RR}}. \quad (58)$$

Substituting this relation and $\phi = M_{\text{pl}}f_{,R}$ into Eq. (57), it follows that

$$G_{\text{eff}} = \frac{G}{f_{,R}} \frac{1 + 4(f_{,RR}/f_{,R})(k/a)^2}{1 + 3(f_{,RR}/f_{,R})(k/a)^2}, \quad \eta = \frac{1 + 2(f_{,RR}/f_{,R})(k/a)^2}{1 + 4(f_{,RR}/f_{,R})(k/a)^2}, \quad (59)$$

which agree with those derived in Ref. [31].

The viable dark energy models based on $f(R)$ theories were constructed to have a large mass M in the deep matter-dominated epoch [43,47,48], i.e. $f_{,RR} \gg 1$ and $f_{,R} \simeq 1$ for $R \gg H_0^2$, where H_0 is the Hubble parameter today. In the regime $(f_{,RR}/f_{,R})(k/a)^2 \ll 1$ (or $M^2 f_{,R} \gg k^2/a^2$) one has $G_{\text{eff}} \simeq G$ and $\eta \simeq 1$, so that the evolution of density perturbations is similar to that in GR. At late times the mass term M gets smaller with the growth of $f_{,RR}$. Since $G_{\text{eff}} \simeq 4G/(3f_{,R})$ and $\eta \simeq 1/2$ for $(f_{,RR}/f_{,R})(k/a)^2 \gg 1$, the growth rate of matter perturbations is larger than that in the Λ CDM model, e.g., $\delta_m \propto t^{(\sqrt{33}-1)/6}$ during the matter-dominated epoch [43,48].

Substituting Eq. (59) into Eq. (55) we obtain $\Phi_{\text{eff}} \simeq -(3/2)(aH/k)^2 \tilde{\Omega}_m \delta_m$, where $\tilde{\Omega}_m \equiv \Omega_m/f_{,R}$. This means that the anisotropic parameter η between the two gravitational potentials practically compensates the modification induced by the gravitational coupling G_{eff} , i.e. $G_{\text{eff}}(1 + \eta)/2 = G/f_{,R}$. During the matter dominance ($\tilde{\Omega}_m \simeq 1$ and $a \propto t^{2/3}$) the evolution of the effective gravitational potential on sub-horizon scales is given by $\Phi_{\text{eff}} \propto t^{(\sqrt{33}-5)/6}$ [43].

5.2. Brans–Dicke theories

Brans–Dicke theories [10] with the field potential $V(\phi)$ correspond to the choice

$$K = \frac{M_{\text{pl}}\omega_{\text{BD}}X}{\phi} - V(\phi), \quad G_3 = 0 = G_5, \quad G_4 = \frac{1}{2}M_{\text{pl}}\phi, \quad (60)$$

where ω_{BD} is the Brans–Dicke parameter (which is constant). Compared to original Brans–Dicke theories we have introduced the reduced Planck mass M_{pl} in K and G_4 such that the field ϕ has a dimension of mass. The difference from $f(R)$ theories appears for the kinetic term $M_{\text{pl}}\omega_{\text{BD}}X/\phi$, in which case $D_9 = -M_{\text{pl}}\omega_{\text{BD}}/\phi$. From Eqs. (45) and (52) it follows that

$$G_{\text{eff}} = \frac{M_{\text{pl}}}{\phi} \frac{4 + 2\omega_{\text{BD}} + 2(\phi/M_{\text{pl}})(Ma/k)^2}{3 + 2\omega_{\text{BD}} + 2(\phi/M_{\text{pl}})(Ma/k)^2} G, \quad \eta = \frac{1 + \omega_{\text{BD}} + (\phi/M_{\text{pl}})(Ma/k)^2}{2 + \omega_{\text{BD}} + (\phi/M_{\text{pl}})(Ma/k)^2}, \quad (61)$$

where

$$M^2 = V_{,\phi\phi} + \frac{\omega_{\text{BD}}M_{\text{pl}}}{\phi^3} [\dot{\phi}^2 - \phi(\ddot{\phi} + 3H\dot{\phi})]. \quad (62)$$

The results (57) in $f(R)$ theories can be recovered by setting $\omega_{\text{BD}} = 0$ in Eq. (61). It is convenient to express M^2 solely in terms of the potential. This can be done by using the scalar field equation of motion, and we obtain $M^2 \simeq V_{,\phi\phi} + V_{,\phi}/\phi$, where we have neglected $\mathcal{O}(M_{\text{pl}}H^2/\phi)$ terms.

In the limit where $\omega_{\text{BD}} \rightarrow \infty$ or $M^2 \rightarrow \infty$ (with $\phi \simeq M_{\text{pl}}$) we recover the General Relativistic behavior: $G_{\text{eff}} \simeq G$ and $\eta \simeq 1$. In the limit that $M^2 \rightarrow 0$ we have $G_{\text{eff}} \simeq (M_{\text{pl}}/\phi)(4 + 2\omega_{\text{BD}})/(3 + 2\omega_{\text{BD}})G$ and $\eta \simeq (1 + \omega_{\text{BD}})/(2 + \omega_{\text{BD}})$ [29,31]. The effective gravitational coupling in the latter case also agrees with the one corresponding to the gravitational force between two test particles [52].

5.3. Kinetic gravity braiding

Let us consider the kinetic gravity braiding [53] described by the Lagrangian

$$K = K(\phi, X), \quad G_3 = G_3(\phi, X), \quad G_4 = \frac{1}{2}M_{\text{pl}}^2, \quad G_5 = 0. \quad (63)$$

Since $A_6 = -2XG_{3,X}$, $B_6 = B_8 = 2M_{\text{pl}}^2$, $B_7 = 0$ in this case, it follows that

$$G_{\text{eff}} = \frac{M^2 - D_9(k/a)^2}{M^2 - (D_9 + 2X^2 G_{3,X}^2/M_{\text{pl}}^2)(k/a)^2} G, \quad \eta = 1, \quad (64)$$

where

$$D_9 = -K_{,X} - 2(G_{3,X} + XG_{3,XX})\ddot{\phi} - 4HG_{3,X}\dot{\phi} + 2G_{3,\phi} - 2XG_{3,\phi X}. \quad (65)$$

In the limit that $M^2 \rightarrow \infty$ we have $G_{\text{eff}} \rightarrow G$, so that the General Relativistic behavior is recovered.

Let us consider the theories in which both K and G_3 depend only on X , i.e. $K = K(X)$ and $G_3 = G_3(X)$. Since $M^2 = 0$, $G_{3,\phi} = 0$, and $G_{3,\phi X} = 0$ in such theories, the effective gravitational coupling is given by

$$G_{\text{eff}} = G \left\{ 1 + \frac{G_{3,X}^2 \dot{\phi}^4}{2M_{\text{pl}}^2 [K_{3,X} + 2(\ddot{\phi} + 2H\dot{\phi})G_{3,X} + G_{3,XX}\dot{\phi}^2\ddot{\phi}] - G_{3,X}^2 \dot{\phi}^4} \right\}. \quad (66)$$

This result agrees with that derived in Ref. [37] in which the authors studied the evolution of perturbations for the functions $K = -X$ and $G_3 \propto X^n$ (which corresponds to the Dvali and Turner model [23]).

One can also extend the analysis to the case where G_4 is a function of ϕ , i.e. $\mathcal{L} = G_4(\phi)R + K(\phi, X) - G_3(\phi, X)\square\phi$. The perturbation equations for the theories with $G_3(\phi, X) = \xi(\phi)X$ were derived in Ref. [36] (see also Refs. [34,35] for specific choices of the functions $G_4(\phi)$ and $K(\phi, X)$).

5.4. Covariant Galileon

The covariant Galileon without the field potential corresponds to [54]

$$K = -c_2 X, \quad G_3 = \frac{c_3}{M^3} X, \quad G_4 = \frac{1}{2} M_{\text{pl}}^2 - \frac{c_4}{M^6} X^2, \quad G_5 = \frac{3c_5}{M^9} X^2, \quad (67)$$

where c_i ($i = 2, 3, 4, 5$) are dimensionless constants and M is the constant having the dimension of mass. For the choice (67) we confirmed that all the coefficients in Eqs. (32)–(35) are equivalent to those given in Ref. [38]. Hence Eqs. (45) and (52) reproduce the effective gravitational coupling G_{eff} and the anisotropic parameter η derived in [38].

For the covariant Galileon there exists a stable de Sitter solution where $X = \text{constant}$ [55] (see also Refs. [56,57] for related works). Since G_{eff} is larger than G before the solution reaches the de Sitter attractor, the growth rate of matter perturbations is larger than that in the Λ CDM model. In addition the variation of the effective gravitational potential Φ_{eff} can be more significant than that in $f(R)$ gravity, because η can be larger than 1 in the early cosmological epoch [38].

5.5. Field derivative couplings with the Einstein tensor

The dark energy model of Gubitosi and Linder [26] corresponds to

$$K = X, \quad G_4 = \frac{1}{2} M_{\text{pl}}^2, \quad G_5 = -\lambda \frac{\phi}{M_{\text{pl}}^2}, \quad (68)$$

where λ is a dimensionless constant (see also Refs. [25,58]). The Lagrangian in Ref. [26] involves the term $(\lambda/M_{\text{pl}}^2)G_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi$, but this is equivalent to $-\lambda(\phi/M_{\text{pl}}^2)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi$ after the integration by parts.

Since $A_6 = -4\lambda H\dot{\phi}M_{\text{pl}}^2$, $B_6 = 2M_{\text{pl}}^2 + 4\lambda X/M_{\text{pl}}^2$, $B_7 = -4\lambda(\ddot{\phi} + H\dot{\phi})/M_{\text{pl}}^2$, $B_8 = 2M_{\text{pl}}^2 - 4\lambda X/M_{\text{pl}}^2$, and $D_9 = -1 - 6\lambda H^2/M_{\text{pl}}^2 - 4\lambda\dot{H}/M_{\text{pl}}^2$, it follows that

$$G_{\text{eff}} = \left(1 + \frac{\xi_1}{\xi_2} \right) G, \quad (69)$$

$$\eta = 1 - \frac{2\lambda[M_{\text{pl}}^2\dot{\phi}^2 + 2\lambda(2H\dot{\phi}\ddot{\phi} + 2\ddot{\phi}^2 + 3H^2\dot{\phi}^2 + 2H\dot{\phi}^2)]}{M_{\text{pl}}^6 + \lambda M_{\text{pl}}^2(6M_{\text{pl}}^2 H^2 + 4M_{\text{pl}}^2 \dot{H} + \dot{\phi}^2) + 2\lambda^2(7H^2\dot{\phi}^2 + 2\dot{H}\dot{\phi}^2 + 4\ddot{\phi}^2 + 8H\dot{\phi}\ddot{\phi})}, \quad (70)$$

where

$$\xi_1 \equiv \lambda[3M_{\text{pl}}^6\dot{\phi}^2 + \lambda M_{\text{pl}}^2(18M_{\text{pl}}^2 H^2\dot{\phi}^2 + 12M_{\text{pl}}^2 \dot{H}\dot{\phi}^2 + 8M_{\text{pl}}^2 \ddot{\phi}^2 - \dot{\phi}^4) + 2\lambda^2\dot{\phi}^3(8H\ddot{\phi} + 9H^2\dot{\phi} - 2\dot{H}\dot{\phi})], \quad (71)$$

$$\xi_2 \equiv M_{\text{pl}}^{10} + 2\lambda M_{\text{pl}}^6(3M_{\text{pl}}^2 H^2 + 2M_{\text{pl}}^2 \dot{H} - \dot{\phi}^2) + \lambda^2 M_{\text{pl}}^2(\dot{\phi}^4 + 16M_{\text{pl}}^2 H\dot{\phi}\ddot{\phi} - 4M_{\text{pl}}^2 H^2\dot{\phi}^2 - 8M_{\text{pl}}^2 \dot{H}\dot{\phi}^2) + 2\lambda^3\dot{\phi}^3(2\dot{H}\dot{\phi} - 8H\ddot{\phi} - 9H^2\dot{\phi}). \quad (72)$$

For $\lambda \neq 0$ one has $G_{\text{eff}} \neq G$ and $\eta \neq 1$, so that the evolution of perturbations is different from that in GR. Note that the results (69) and (70) are derived for the first time in this Letter.

6. Conclusions

In this Letter we have derived the full equations of scalar density perturbations for the perturbed metric (21). The Newtonian gauge corresponds to the choice $\chi = 0$, which fixes the temporal part of the gauge-transformation vector. Since the different gauge choices (such as $\delta\phi = 0$) are possible, our linear perturbation equations can be applied to other gauges as well.

For the perturbations deep inside the Hubble radius the quasi-static approximation employed in Section 4 is accurate for an effectively massless scalar field. In fact this is the case for kinetic gravity braidings and covariant Galileons. In the dark energy models based on $f(R)$ theories and Brans–Dicke theories, the mass of the scalar field degree of freedom needs to be large in the region of high density for consistency with local gravity constraints. In order to accommodate such cases we have taken into account the effective mass M in estimating the effective gravitational coupling G_{eff} . For the quasi-static approximation to work it is necessary that the time-derivatives of the field perturbation $\delta\phi$ are neglected relative to the terms including $c_s^2(k^2/a^2)\delta\phi$, where c_s is the scalar propagation speed whose explicit expression is given in Ref. [46]. This implies that the perturbation field cannot be fast oscillating. In other words, provided that the oscillating mode of the field perturbation is suppressed relative to the matter-induced mode, the quasi-static approximation on sub-horizon scales can be trustable.

In order to estimate the growth rate of perturbations relevant to large-scale structure and weak lensing, it is sufficient to use the approximate results of (45) and (55) with the anisotropic parameter η given by Eq. (52). We applied our formulas to a number of modified gravitational models of dark energy and found that they nicely reproduce the previously known results. There are some new models – such as the field derivative couplings with the Einstein tensor – in which the evolution of perturbations deserves for further detailed investigation.

For the large-scale perturbations associated with the integrated-Sachs–Wolfe effect in the CMB anisotropies the sub-horizon approximation in Section 4 is no longer valid. Instead we need to integrate the perturbation equations (26)–(31) numerically, along the lines of Ref. [38]. It will be of interest how the joint data analysis of CMB combined with the observations of large-scale structure and weak lensing place constraints on each modified gravitational model of dark energy accommodated by the general action (6).

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Appendix A. Coefficients relevant to sub-horizon perturbations

Here we write the explicit forms of the coefficients A_6 , B_6 , B_7 , B_8 , and D_9 :

$$A_6 = -2XG_{3,X} - 4H(G_{4,X} + 2XG_{4,XX})\dot{\phi} + 2G_{4,\phi} + 4XG_{4,\phi X} + 4H(G_{5,\phi} + XG_{5,\phi X})\dot{\phi} - 2H^2X(3G_{5,X} + 2XG_{5,XX}), \quad (\text{A.1})$$

$$B_6 = 4[G_4 - X(\ddot{\phi}G_{5,X} + G_{5,\phi})], \quad (\text{A.2})$$

$$B_7 = -4G_{4,X}H\dot{\phi} - 4(G_{4,X} + 2XG_{4,XX})\ddot{\phi} + 4G_{4,\phi} - 8XG_{4,\phi X} + 4(G_{5,\phi} + XG_{5,\phi X})\ddot{\phi} - 4H[(G_{5,X} + XG_{5,XX})\ddot{\phi} - G_{5,\phi} + XG_{5,\phi X}]\dot{\phi} + 4X[G_{5,\phi\phi} - (H^2 + \dot{H})G_{5,X}], \quad (\text{A.3})$$

$$B_8 = 4[G_4 - 2XG_{4,X} - X(H\dot{\phi}G_{5,X} - G_{5,\phi})], \quad (\text{A.4})$$

$$D_9 = -K_{,X} - 2(G_{3,X} + XG_{3,XX})\ddot{\phi} - 4HG_{3,X}\dot{\phi} + 2G_{3,\phi} - 2XG_{3,\phi X} + [-4H(3G_{4,XX} + 2XG_{4,XXX})\ddot{\phi} + 4H(3G_{4,\phi X} - 2XG_{4,\phi XX})]\dot{\phi} + (6G_{4,\phi X} + 4XG_{4,\phi XX})\ddot{\phi} - 20H^2XG_{4,XX} + 4XG_{4,\phi\phi X} - 4\dot{H}(G_{4,X} + 2XG_{4,XX}) - 6H^2G_{4,X} + \{4H(2G_{5,\phi X} + XG_{5,\phi XX})\ddot{\phi} - 4H[(H^2 + \dot{H})(G_{5,X} + XG_{5,XX}) - XG_{5,\phi\phi X}]\}\dot{\phi} - 4H^2X^2G_{5,\phi XX} - 2H^2(G_{5,X} + 5XG_{5,XX} + 2X^2G_{5,XXX})\ddot{\phi} + 2(3H^2 + 2\dot{H})G_{5,\phi} + 4\dot{H}XG_{5,\phi X} + 10H^2XG_{5,\phi X}. \quad (\text{A.5})$$

References

- [1] A.G. Riess, et al., *Astron. J.* 116 (1998) 1009; A.G. Riess, et al., *Astron. J.* 117 (1999) 707; S. Perlmutter, et al., *Astrophys. J.* 517 (1999) 565.
- [2] D.N. Spergel, et al., *Astrophys. J. Suppl.* 148 (2003) 175; E. Komatsu, et al., WMAP Collaboration, *Astrophys. J. Suppl.* 192 (2011) 18.
- [3] D.J. Eisenstein, et al., SDSS Collaboration, *Astrophys. J.* 633 (2005) 560; W.J. Percival, et al., *Mon. Not. Roy. Astron. Soc.* 401 (2010) 2148.
- [4] S. Weinberg, *Rev. Mod. Phys.* 61 (1989) 1.
- [5] V. Sahni, A.A. Starobinsky, *Int. J. Mod. Phys. D* 9 (2000) 373; S.M. Carroll, *Living Rev. Rel.* 4 (2001) 1; T. Padmanabhan, *Phys. Rept.* 380 (2003) 235; P.J.E. Peebles, B. Ratra, *Rev. Mod. Phys.* 75 (2003) 559.

- [6] E.J. Copeland, M. Sami, S. Tsujikawa, *Int. J. Mod. Phys. D* 15 (2006) 1753.
- [7] Y. Fujii, *Phys. Rev. D* 26 (1982) 2580;
L.H. Ford, *Phys. Rev. D* 35 (1987) 2339;
C. Wetterich, *Nucl. Phys. B* 302 (1988) 668;
B. Ratra, P.J.E. Peebles, *Phys. Rev. D* 37 (1988) 3406;
R.R. Caldwell, R. Dave, P.J. Steinhardt, *Phys. Rev. Lett.* 80 (1998) 1582.
- [8] C. Armendariz-Picon, T. Damour, V.F. Mukhanov, *Phys. Lett. B* 458 (1999) 209;
T. Chiba, T. Okabe, M. Yamaguchi, *Phys. Rev. D* 62 (2000) 023511;
C. Armendariz-Picon, V.F. Mukhanov, P.J. Steinhardt, *Phys. Rev. Lett.* 85 (2000) 4438.
- [9] S. Capozziello, *Int. J. Mod. Phys. D* 11 (2002) 483;
S. Capozziello, S. Carloni, A. Troisi, *Recent Res. Dev. Astron. Astrophys.* 1 (2003) 625;
S. Capozziello, V.F. Cardone, S. Carloni, A. Troisi, *Int. J. Mod. Phys. D* 12 (2003) 1969;
S.M. Carroll, V. Duvvuri, M. Trodden, M.S. Turner, *Phys. Rev. D* 70 (2004) 043528.
- [10] C. Brans, R.H. Dicke, *Phys. Rev.* 124 (1961) 925.
- [11] G.R. Dvali, G. Gabadadze, M. Porrati, *Phys. Lett. B* 485 (2000) 208.
- [12] A. Nicolis, R. Rattazzi, E. Trincherini, *Phys. Rev. D* 79 (2009) 064036.
- [13] T.P. Sotiriou, V. Faraoni, *Rev. Mod. Phys.* 82 (2010) 451;
A. De Felice, S. Tsujikawa, *Living Rev. Rel.* 13 (2010) 3;
S. Tsujikawa, *Lecture Notes in Phys.* 800 (2010) 99.
- [14] J. Khoury, A. Weltman, *Phys. Rev. Lett.* 93 (2004) 171104;
J. Khoury, A. Weltman, *Phys. Rev. D* 69 (2004) 044026.
- [15] J.A.R. Cembranos, *Phys. Rev. D* 73 (2006) 064029;
I. Navarro, K. Van Acoleyen, *JCAP* 0702 (2007) 022;
T. Faulkner, M. Tegmark, E.F. Bunn, Y. Mao, *Phys. Rev. D* 76 (2007) 063505;
S. Capozziello, S. Tsujikawa, *Phys. Rev. D* 77 (2008) 107501.
- [16] C. Deffayet, G.R. Dvali, G. Gabadadze, A.I. Vainshtein, *Phys. Rev. D* 65 (2002) 044026;
M. Porrati, *Phys. Lett. B* 534 (2002) 209;
M.A. Luty, M. Porrati, R. Rattazzi, *JHEP* 0309 (2003) 029;
R. Gannouji, M. Sami, *Phys. Rev. D* 82 (2010) 024011.
- [17] A.I. Vainshtein, *Phys. Lett. B* 39 (1972) 393.
- [18] M. Ostrogradski, *Mem. Ac. St. Petersburg VI* 4 (1850) 385.
- [19] G.W. Horndeski, *Int. J. Theor. Phys.* 10 (1974) 363.
- [20] C. Deffayet, X. Gao, D.A. Steer, G. Zahariade, arXiv:1103.3260 [hep-th].
- [21] T. Kobayashi, M. Yamaguchi, J.'i. Yokoyama, *Progr. Theoret. Phys.* 126 (2011) 511, arXiv:1105.5723 [hep-th].
- [22] C. Charmousis, E.J. Copeland, A. Padilla, P.M. Saffin, arXiv:1106.2000 [hep-th].
- [23] G. Dvali, M.S. Turner, astro-ph/0301510.
- [24] S.M. Carroll, et al., *Phys. Rev. D* 71 (2005) 063513;
S. Nojiri, S.D. Odintsov, M. Sasaki, *Phys. Rev. D* 71 (2005) 123509;
G. Calcagni, S. Tsujikawa, M. Sami, *Classical Quantum Gravity* 22 (2005) 3977;
A. De Felice, M. Hindmarsh, M. Trodden, *JCAP* 0608 (2006) 005.
- [25] C. Germani, A. Kehagias, *Phys. Rev. Lett.* 105 (2010) 011302;
C. Germani, A. Kehagias, *Phys. Rev. Lett.* 106 (2011) 161302.
- [26] G. Gubitosi, E.V. Linder, arXiv:1106.2815 [astro-ph.CO].
- [27] B. Boisseau, G. Esposito-Farese, D. Polarski, A.A. Starobinsky, *Phys. Rev. Lett.* 85 (2000) 2236;
G. Esposito-Farese, D. Polarski, *Phys. Rev. D* 63 (2001) 063504.
- [28] A. Riazuelo, J.P. Uzan, *Phys. Rev. D* 66 (2002) 023525.
- [29] V. Acquaviva, L. Verde, *JCAP* 0712 (2007) 001.
- [30] Y.S. Song, L. Hollenstein, G. Caldera-Cabral, K. Koyama, *JCAP* 1004 (2010) 018.
- [31] S. Tsujikawa, *Phys. Rev. D* 76 (2007) 023514.
- [32] P. Zhang, *Phys. Rev. D* 73 (2006) 123504;
S.M. Carroll, I. Sawicki, A. Silvestri, M. Trodden, *New J. Phys.* 8 (2006) 323;
R. Bean, D. Bernat, L. Pogosian, A. Silvestri, M. Trodden, *Phys. Rev. D* 75 (2007) 064020;
Y.S. Song, W. Hu, I. Sawicki, *Phys. Rev. D* 75 (2007) 044004;
T. Faulkner, M. Tegmark, E.F. Bunn, Y. Mao, *Phys. Rev. D* 76 (2007) 063505.
- [33] N. Chow, J. Khoury, *Phys. Rev. D* 80 (2009) 024037.
- [34] F.P. Silva, K. Koyama, *Phys. Rev. D* 80 (2009) 121301.
- [35] T. Kobayashi, H. Tashiro, D. Suzuki, *Phys. Rev. D* 81 (2010) 063513;
T. Kobayashi, *Phys. Rev. D* 81 (2010) 103533.
- [36] A. De Felice, S. Mukohyama, S. Tsujikawa, *Phys. Rev. D* 82 (2010) 023524.
- [37] R. Kimura, K. Yamamoto, *JCAP* 1104 (2011) 025.
- [38] A. De Felice, R. Kase, S. Tsujikawa, *Phys. Rev. D* 83 (2011) 043515.
- [39] M. Ishak, A. Upadhye, D.N. Spergel, *Phys. Rev. D* 74 (2006) 043513;
L. Knox, Y.S. Song, J.A. Tyson, *Phys. Rev. D* 74 (2006) 023512;
D. Huterer, E.V. Linder, *Phys. Rev. D* 75 (2007) 023519;
P. Zhang, et al., *Phys. Rev. Lett.* 99 (2007) 141302;
S. Wang, L. Hui, M. May, Z. Haiman, *Phys. Rev. D* 76 (2007) 063503;
B. Jain, P. Zhang, *Phys. Rev. D* 78 (2008) 063503;
S.F. Daniel, et al., *Phys. Rev. D* 77 (2008) 103513;
E. Bertschinger, P. Zukin, *Phys. Rev. D* 78 (2008) 024015;
J.B. Dent, S. Dutta, L. Perivolaropoulos, *Phys. Rev. D* 80 (2009) 023514;
Y.S. Song, K. Koyama, *JCAP* 0901 (2009) 048;
Y.S. Song, O. Dore, *JCAP* 0903 (2009) 025;
G.B. Zhao, et al., *Phys. Rev. D* 79 (2009) 083513;
S.A. Thomas, F.B. Abdalla, J. Weller, *Mon. Not. Roy. Astron. Soc.* 395 (2009) 197;
T. Narikawa, K. Yamamoto, *Phys. Rev. D* 81 (2010) 043528;
R. Bean, M. Tangmatitham, *Phys. Rev. D* 81 (2010) 083534.
- [40] J.M. Bardeen, *Phys. Rev. D* 22 (1980) 1882.
- [41] X. Gao, D.A. Steer, arXiv:1107.2642 [astro-ph.CO].
- [42] A. De Felice, S. Tsujikawa, *Phys. Rev. D* 84 (2011) 083504.

- [43] S. Tsujikawa, Phys. Rev. D 77 (2008) 023507.
- [44] S. Tsujikawa, K. Uddin, S. Mizuno, R. Tavakol, J. Yokoyama, Phys. Rev. D 77 (2008) 103009.
- [45] A.A. Starobinsky, JETP Lett. 68 (1998) 757.
- [46] A. De Felice, S. Tsujikawa, arXiv:1110.3878 [gr-qc].
- [47] W. Hu, I. Sawicki, Phys. Rev. D 76 (2007) 064004;
S.A. Appleby, R.A. Battye, Phys. Lett. B 654 (2007) 7;
L. Amendola, S. Tsujikawa, Phys. Lett. B 660 (2008) 125.
- [48] A.A. Starobinsky, JETP Lett. 86 (2007) 157.
- [49] S. Tsujikawa, K. Uddin, R. Tavakol, Phys. Rev. D 77 (2008) 043007;
S.A. Appleby, R.A. Battye, JCAP 0805 (2008) 019;
A. de la Cruz-Dombriz, A. Dobado, A.L. Maroto, Phys. Rev. D 77 (2008) 123515;
H. Motohashi, A.A. Starobinsky, J. Yokoyama, Int. J. Mod. Phys. D 18 (2009) 1731.
- [50] C. Schmid, J.P. Uzan, A. Riazuelo, Phys. Rev. D 71 (2005) 083512;
L. Amendola, M. Kunz, D. Sapone, JCAP 0804 (2008) 013.
- [51] J. O'Hanlon, Phys. Rev. Lett. 29 (1972) 137;
T. Chiba, Phys. Lett. B 575 (2003) 1.
- [52] Y. Fujii, K.i. Maeda, The Scalar-Tensor Theory of Gravitation, Cambridge University Press, 2003, p. 47.
- [53] C. Deffayet, O. Pujolas, I. Sawicki, A. Vikman, JCAP 1010 (2010) 026.
- [54] C. Deffayet, G. Esposito-Farese, A. Vikman, Phys. Rev. D 79 (2009) 084003;
C. Deffayet, S. Deser, G. Esposito-Farese, Phys. Rev. D 80 (2009) 064015.
- [55] A. De Felice, S. Tsujikawa, Phys. Rev. Lett. 105 (2010) 111301, arXiv:1008.4236 [hep-th].
- [56] T. Kobayashi, M. Yamaguchi, J. Yokoyama, Phys. Rev. Lett. 105 (2010) 231302.
- [57] C. Burrage, C. de Rham, D. Seery, A.J. Tolley, JCAP 1101 (2011) 014.
- [58] L. Amendola, Phys. Lett. B 301 (1993) 175.