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Slow-roll, acceleration, the big rip and the Wentzel–Kramers–Brillouin approximation in the non-linear Schrödinger-type formulation of scalar field cosmology

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Abstract. Aspects of the non-linear Schrödinger (NLS) type of formulation of scalar (phantom) field cosmology for slow-roll, acceleration, the Wentzel–Kramers–Brillouin (WKB) approximation and the big rip singularity are presented. Slow-roll parameters for the curvature and barotropic density terms are introduced. We re-express all slow-roll parameters, slow-roll conditions and the acceleration condition in NLS form. The WKB approximation in the NLS formulation is also discussed while simplifying to the linear case. Most of the Schrödinger potentials in the NLS formulation are very slowly varying; hence the WKB approximation is valid in some ranges. In the NLS form of the big rip singularity, two quantities are infinite instead of three. We also found that approaching the big rip, $w_{\text{eff}} \to -1 + 2/3q$ ($q < 0$), which is the same as the effective phantom equation of state in the flat case.

Keywords: dark energy theory, inflation

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1. Introduction

Cosmology with a scalar field is one of today's main research aims. Although the scalar field has not yet been observed, it is motivated from many ideas in high energy physics and quantum gravities. Near future TeV scale experiments at LHC and Tevatron might discover its existence. It has been widely accepted in theoretical frameworks, especially in model building of contemporary cosmology, that the field sources acceleration expansion at early time, i.e. inflation, in order to solve horizon and flatness problems [1] and it also plays a similar role in explaining present acceleration observed and confirmed from the cosmic microwave background [2], large scale structure surveys [3] and supernovae of type Ia [4]–[6]. In the late acceleration, it plays the role of dark energy (see [7] for reviews). Both inflation and acceleration appear convincing from recent combined results [8], with the possibility that the scalar field could be a phantom, i.e. having an equation of state coefficient $w_\phi < -1$. The phantom equation of state is attained from negative kinetic energy terms in the Lagrangian density [9, 10]. Using the BBN constraint of the limit of the expansion rate [11, 12] with the most recent WMAP five-year result [13], $w_{\phi,0} = -1.09 \pm 0.12$ at 68% CL. The WMAP five-year result combined with baryon acoustic oscillation from the large scale structure survey (from SDSS and 2dFGRS) [14] and type Ia supernovae data (from HST [5], SNLS [6] and ESSENCE [15]), assuming dynamical w with a flat universe, yields $-1.38 < w_{\phi,0} < -0.86$ at 95% CL and $w_{\phi,0} = -1.12 \pm 0.13$ at 68% CL. Although the phantom field has room for observation, in a flat universe the idea suffers from an unwanted big rip singularity [16, 17]. However there have been many attempts to resolve the singularity from both phenomenological and fundamental inspiration [18].

Inflationary models in the presence of other fields behaving in a barotropic-like way apart from having only a single scalar field were considered, such as that in [19] where the scale invariant spectrum in the cosmic microwave background was claimed to be generated not from fluctuation of the scalar field alone but rather from both a scalar field and interaction between gravity and other gauge fields such as Dirac and gauge vector fields. This is similar to the situation in the late universe in which the acceleration happens in the presence of both dark matter fluid and scalar fluid (as dark energy). Proposals of mathematical alternatives to the standard Friedmann canonical scalar field cosmology with barotropic perfect fluid were advanced, such as the non-linear Ermakov–Pinney equation [20, 21]. There are also other applications of the Ermakov–Pinney equation; for example in [22], a link from standard cosmology with $k > 0$ in the Ermakov system to Bose–Einstein condensates was shown. Another example is a connection from the generalized Ermakov–Pinney equation with a perturbative scheme to the generalized WKB method of comparison equations [23]. Recently a link from standard canonical scalar field cosmology in the Friedmann–Lemaître–Robertson–Walker (FLRW) background with barotropic fluid to quantum mechanics was established. This was realized from the fact that solutions of the generalized Ermakov–Pinney equation correspond to solutions of the non-linear Schrödinger-type equation, hereafter the NLS equation [21, 24]. A connection from the NLS-type formulation to the Friedmann scalar field cosmology formulation is concluded in [25] where standard cosmological quantities are reinterpreted in the language of quantum mechanics assuming power-law expansion, $a \sim t^q$, and the phantom field case is included. The quantities in the new form satisfy a non-linear Schrödinger-type equation. In most circumstances, the scalar field exact solution $\phi(t)$ can be solved analytically only when assuming flat geometry ($k = 0$) and scalar field fluid domination. When $k \neq 0$ with more than one fluid component, it is not always possible to solve the system analytically in the standard Friedmann formulation. Transforming standard Friedmann cosmological quantities into NLS forms could help in obtaining the solution [26, 27]. In the NLS formulation, the independent variable t in the standard formulation is re-scaled to the variable x. However, pre-knowledge of the scale factor as a function of time, $a(t)$, must be assumed in order to express NLS quantities. It is interesting to see the other features of the field velocity, $\dot{\phi}$, e.g. the acceleration condition and slow-roll approximation, written in the NLS formulation. Mathematical tools such as the WKB approximation in quantum mechanics might also be interesting for application in standard scalar field cosmology. It is worthwhile to investigate this possibility. It is worth noting that a Schrödinger-type equation in scalar field cosmology was previously considered in a different procedure for studying inflation and phantom field problems [28].

We introduce the NLS formulation in section 2. The slow-roll conditions in both formulations are discussed in section 3 where we define slow-roll parameters for barotropic fluid and curvature terms. Then in section 4 we show acceleration conditions in NLS form. The WKB approximation is performed in section 5. The NLS form of the big rip singularity is in section 6 and finally conclusions are drawn in section 7.

2. Scalar field cosmology in the NLS formulation

Two perfect fluids are considered in an FLRW universe: barotropic fluid and the scalar field. The barotropic equation of state is $p_\gamma = w_\gamma \rho_\gamma$ with w_γ expressed as n where

$n = 3(1 + w_\gamma)$. The scalar field pressure obeys $p_\phi = w_\phi \rho_\phi$. The total density and pressure of the mixture are sums of the two components. The evolution of the barotropic density is governed by a conservation equation, $\dot{\rho}_\gamma = -nH\rho_\gamma$, with solution $\rho_\gamma = D/a^n$, where a is a scale factor, the dot denotes the time derivative, and $D \geq 0$ is a proportionality constant. Using the scalar field Lagrangian density, $\mathcal{L} = (1/2)\epsilon\dot{\phi}^2 - V(\phi)$, i.e. minimally coupling to gravity,

$$\rho_\phi = \tfrac{1}{2}\epsilon\dot{\phi}^2 + V(\phi), \qquad p_\phi = \tfrac{1}{2}\epsilon\dot{\phi}^2 - V(\phi). \tag{1}$$

The branch $\epsilon = 1$ is for the non-phantom case and $\epsilon = -1$ is for the phantom case [17]. Note that the phantom behaviour ($\rho_\phi < -p_\phi$) can also be obtained in the case of non-minimal coupling to gravity [29]. The dynamics of the field is controlled by the conservation equation

$$\epsilon\left(\ddot{\phi} + 3H\dot{\phi}\right) = -\frac{\mathrm{d}V}{\mathrm{d}\phi}. \tag{2}$$

The spatial expansion of the universe sources friction to dynamics of the field in equation (2) via the Hubble parameter H. The Hubble parameter is governed by the Friedmann equation,

$$H^2 = \frac{\kappa^2}{3}\rho_{\rm tot} - \frac{k}{a^2}, \tag{3}$$

where here $\rho_{\rm tot} = (1/2)\epsilon\dot{\phi}^2 + V + D/a^n$, and by the acceleration equation,

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}(\rho_{\rm tot} + 3p_{\rm tot}), \tag{4}$$

which does not depend on k. This gives the acceleration condition

$$p_{\rm tot} < -\frac{\rho_{\rm tot}}{3}. \tag{5}$$

Here $p_{\rm tot} = w_{\rm eff}\rho_{\rm tot}$, $\kappa^2 \equiv 8\pi G = 1/M_{\rm P}^2$, G is Newton's gravitational constant, $M_{\rm P}$ is the reduced Planck mass, k is the spatial curvature and $w_{\rm eff} = (\rho_\phi w_\phi + \rho_\gamma w_\gamma)/\rho_{\rm tot}$. Using these facts, it is straightforward to show that

$$\epsilon\dot{\phi}(t)^2 = -\frac{2}{\kappa^2}\left[\dot{H} - \frac{k}{a^2}\right] - \frac{nD}{3a^n}, \tag{6}$$

$$V(\phi) = \frac{3}{\kappa^2}\left[H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right] + \left(\frac{n-6}{6}\right)\frac{D}{a^n}. \tag{7}$$

The Friedmann formulation of scalar field cosmology above can be transformed to the NLS formulation as one defines NLS quantities [24]:

$$u(x) \equiv a(t)^{-n/2}, \tag{8}$$

$$E \equiv -\frac{\kappa^2 n^2}{12}D, \tag{9}$$

$$P(x) \equiv \frac{\kappa^2 n}{4}a(t)^n\epsilon\dot{\phi}(t)^2. \tag{10}$$

In the NLS formulation, there are no equations analogous to the Friedmann equation or the fluid equation since both of them are written together in the form of a non-linear Schrödinger-type equation[1]:

$$u''(x) + [E - P(x)]\, u(x) = -\frac{nk}{2} u(x)^{(4-n)/n}, \tag{11}$$

where $'$ denotes $\mathrm{d}/\mathrm{d}x$. The independent variable t is scaled to an NLS-independent variable x as $x = \sigma(t)$, such that

$$\dot{x}(t) = u(x), \tag{12}$$

$$\phi(t) = \psi(x). \tag{13}$$

Using equation (10) and $\epsilon\dot{\phi}(t)^2 = \epsilon\dot{x}^2\psi'(x)^2$, we get [25]

$$\epsilon\psi'(x)^2 = \frac{4}{\kappa^2 n} P(x), \tag{14}$$

and hence

$$\psi(x) = \pm\frac{2}{\kappa\sqrt{n}} \int \sqrt{\frac{P(x)}{\epsilon}}\, \mathrm{d}x. \tag{15}$$

The inverse function $\psi^{-1}(x)$ exists for $P(x) \neq 0$ and $n \neq 0$. In these circumstances, $x(t) = \psi^{-1} \circ \phi(t)$ and the scalar field potential, $V \circ \sigma^{-1}(x)$, and $\epsilon\dot{\phi}(t)^2$ can be expressed in the NLS formulation as

$$\epsilon\dot{\phi}(x)^2 = \frac{4}{\kappa^2 n} uu'' + \frac{2k}{\kappa^2} u^{4/n} + \frac{4E}{\kappa^2 n} u^2 = \frac{4P}{\kappa^2 n} u^2, \tag{16}$$

$$V(x) = \frac{12}{\kappa^2 n^2}(u')^2 - \frac{2P}{\kappa^2 n} u^2 + \frac{12E}{\kappa^2 n^2} u^2 + \frac{3k}{\kappa^2} u^{4/n}. \tag{17}$$

From equations (16) and (17), we can find

$$\rho_\phi = \frac{12}{\kappa^2 n^2}(u')^2 + \frac{12E}{\kappa^2 n^2} u^2 + \frac{3k}{\kappa^2} u^{4/n}, \tag{18}$$

$$p_\phi = -\frac{12}{\kappa^2 n^2}(u')^2 + \frac{4P}{\kappa^2 n} u^2 - \frac{12E}{\kappa^2 n^2} u^2 - \frac{3k}{\kappa^2} u^{4/n}. \tag{19}$$

We know that $\rho_\gamma = Du^2 = -12Eu^2/(\kappa^2 n^2)$ from equation (9) and the barotropic pressure is $p_\gamma = [(n-3)/3]\rho_\gamma$; therefore

$$\rho_{\mathrm{tot}} = \frac{12}{\kappa^2 n^2}(u')^2 + \frac{3k}{\kappa^2} u^{4/n}, \tag{20}$$

$$p_{\mathrm{tot}} = -\frac{12}{\kappa^2 n^2}(u')^2 + \frac{4u^2}{\kappa^2 n}[P - E] - \frac{3k}{\kappa^2} u^{4/n}. \tag{21}$$

Using the Schrödinger-type equation (11), then

$$p_{\mathrm{tot}} = -\frac{12}{\kappa^2 n^2}(u')^2 + \frac{4}{\kappa^2 n} uu'' - \frac{k}{\kappa^2} u^{4/n}. \tag{22}$$

[1] The NLS equation considered here is only dependent on x; hence it is not a partial differential equation with a localized soliton-like solution as in [30].

3. Slow-roll conditions

3.1. Slow-roll conditions: flat geometry and scalar field domination

In a flat universe with scalar field domination ($k=0, \rho_\gamma=0$), the Friedmann equation $H^2=\kappa^2\rho_\phi/3$, together with equation (2), yields $\dot{H}=-\kappa^2\dot{\phi}^2\epsilon/2$. For $\epsilon=-1$, we get $\dot{H}>0$ and

$$0<aH^2<\ddot{a}, \tag{23}$$

i.e. the acceleration is greater than the speed of expansion per Hubble radius, $\dot{a}/cH^{-1}$. On the other hand, for $\epsilon=1$, we get $\dot{H}<0$ and

$$0<\ddot{a}<aH^2. \tag{24}$$

The slow-roll condition in [31, 32] assumes a negligible kinetic term; hence $|\epsilon\dot{\phi}^2/2|\ll V(\phi)$, and therefore $\rho_\phi\simeq V(\phi)$. Hence $H^2\simeq\kappa^2V/3$. With this approximation,

$$H^2=-\frac{\dot{H}}{3}+\frac{\kappa^2}{3}V, \quad\Rightarrow\quad H^2\simeq-\frac{\dot{H}}{3}+H^2. \tag{25}$$

This results in an approximation $|\dot{H}|\ll H^2$ from which the slow-roll parameter

$$\varepsilon\equiv-\frac{\dot{H}}{H^2} \tag{26}$$

is defined. Then the condition $|\epsilon\dot{\phi}^2/2|\ll V(\phi)$ is equivalent to $|\varepsilon|\ll 1$, i.e. $-1\ll\varepsilon<0$ for the phantom field case and $0<\varepsilon\ll 1$ for the non-phantom field case. For the non-phantom field, this condition is necessary for inflation to happen (though not sufficient) [32], but for the phantom field case, the slow-roll condition is not needed because the negative kinetic term results in acceleration with $w_\phi\leq-1$. The other slow-roll parameter is defined by balancing the magnitudes of the field friction and acceleration terms in equation (2). This is independent of k or ρ_γ. When friction dominates, $|\ddot{\phi}|\ll|3H\dot{\phi}|$, then

$$\eta\equiv-\frac{\ddot{\phi}}{H\dot{\phi}} \tag{27}$$

is defined [32]. The condition is then $|\eta|\ll 1$ and the fluid equation is approximated to $\dot{\phi}\simeq-V_\phi/3\epsilon H$ which allows the field to roll up the hill when $\epsilon=-1$. Using two conditions, e.g. $|\epsilon\dot{\phi}^2/2|\ll V$ and $|\ddot{\phi}|\ll|3H\dot{\phi}|$, together, one can derive $\varepsilon=(1/2\kappa^2\epsilon)(V_\phi/V)^2$ and $\eta=(1/\kappa^2)(V_{\phi\phi}/V)$ as is well known.

3.2. Slow-roll conditions: non-flat geometry and non-negligible barotropic density

3.2.1. The Friedmann formulation. When considering the case of $k\neq0$ and $\rho_\gamma\neq0$, then

$$\dot{H}=-\frac{\kappa^2}{2}\dot{\phi}^2\epsilon+\frac{k}{a^2}-\frac{n\kappa^2}{6}\frac{D}{a^n}. \tag{28}$$

We can then write the slow-roll condition as $|\kappa^2\epsilon\dot{\phi}^2/6|\ll(\kappa^2V/3)-(k/a^2)+(\kappa^2D/3a^n)$ and hence $H^2\simeq(\kappa^2V/3)+(\kappa^2D/3a^n)-(k/a^2)$. Using this approximation and equation (28) in (3),

$$H^2 \simeq -\frac{\dot{H}}{3} + \frac{k}{3a^2} - \frac{n\kappa^2}{18}\frac{D}{a^n} + H^2, \tag{29}$$

which implies $|-(\dot{H}/3) + (k/3a^2) - (n\kappa^2 D/18a^n)| \ll H^2$. We can re-express this slow-roll condition as

$$|\varepsilon + \varepsilon_k + \varepsilon_D| \ll 1, \tag{30}$$

where $\varepsilon_k \equiv k/a^2H^2$ and $\varepsilon_D \equiv -n\kappa^2 D/6a^n H^2$. Another slow-roll parameter, η, is defined as $\eta \equiv -\ddot{\phi}/H\dot{\phi}$, i.e. the same as for the flat scalar field dominated case since the condition $|\ddot{\phi}| \ll |3H\dot{\phi}|$ is derived from the fluid equation of the field which is independent of k and ρ_γ.

3.2.2. The NLS formulation. In the NLS formulation, the Hubble parameter takes the form

$$H = -\frac{2}{n}u', \tag{31}$$

with

$$\dot{H} = -\frac{2}{n}uu'' = \frac{2}{n}u^2\left[E - P(x)\right] + ku^{4/n}. \tag{32}$$

The slow-roll condition $|\epsilon\dot{\phi}^2/2| \ll V$ using equations (10) and (17) in NLS form is then

$$|P(x)| \ll \frac{3}{n}\left[\left(\frac{u'}{u}\right)^2 + E\right] + \frac{3}{4}knu^{(4-2n)/n}. \tag{33}$$

If the absolute sign is not used, the condition is then $\epsilon\dot{\phi}^2/2 \ll V$, allowing fast-roll negative kinetic energy. Then equation (33), when combined with the NLS equation (11), yields

$$u'' \ll \frac{3}{n}\frac{u'^2}{u} + \left(\frac{3}{n} - 1\right)Eu + \frac{kn}{4}u^{(4-n)/n}. \tag{34}$$

The Friedmann formulation analogue of this condition can be obtained simply by using equations (6) and (7) in the condition. Consider another aspect of slow-roll in the fluid equation; the field acceleration can be written in NLS form:

$$\ddot{\phi} = \frac{2Puu' + P'u^2}{\kappa\sqrt{P\epsilon n}}, \tag{35}$$

while the friction term in NLS form is

$$3H\dot{\phi} = -\frac{12u'u}{n\kappa}\sqrt{\frac{P}{\epsilon n}}. \tag{36}$$

The second slow-roll condition, $|\ddot{\phi}| \ll |3H\dot{\phi}|$, hence corresponds to

$$\left|\frac{P'}{P}\right| \ll \left|-2\left(\frac{6+n}{n}\right)\frac{u'}{u}\right|. \tag{37}$$

This condition yields the approximation $3H\epsilon\dot{\phi}^2 \simeq -\mathrm{d}V/\mathrm{d}\phi$. Using equations (16), (17), (31) and (32), one can express the approximation, $3H\epsilon\dot{\phi}^2 \simeq -\mathrm{d}V/\mathrm{d}\phi$, in NLS form as

$$\frac{P'}{P} \simeq -\frac{2u'}{u} = nHa^{n/2} \tag{38}$$

and finally the slow-roll parameters ε, ε_k and ε_D, introduced previously, become

$$\varepsilon = \frac{nuu''}{2u'^2}, \qquad \varepsilon_k = \frac{n^2ku^{4/n}}{4u'^2}, \qquad \varepsilon_D = \frac{nE}{2}\left(\frac{u}{u'}\right)^2, \tag{39}$$

in NLS form. With the help of NLS equation (11), the sum of the slow-roll parameters takes a simple form:

$$\varepsilon_{\mathrm{tot}} = \varepsilon + \varepsilon_k + \varepsilon_D = \frac{n}{2}\left(\frac{u}{u'}\right)^2 P(x). \tag{40}$$

Finally the slow-roll condition $|\varepsilon_{\mathrm{tot}}| \ll 1$ (equation (30)), in NLS form, is

$$\left|\left(\frac{u}{u'}\right)^2 P(x)\right| \ll 1. \tag{41}$$

Another slow-roll parameter, $\eta = -\ddot{\phi}/H\dot{\phi}$, can be found as follows. First considering $\psi(x) = \phi(t)$ (equation (13)), using the relation $\mathrm{d}/\mathrm{d}t = \dot{x}\,\mathrm{d}/\mathrm{d}x$ and equation (31), we obtain

$$\eta = \frac{n}{2}\left(\frac{u}{u'}\frac{\psi''}{\psi'} + 1\right). \tag{42}$$

Equation (15) yields

$$\psi' = \pm\frac{2}{\kappa}\sqrt{\frac{P}{n\epsilon}} \qquad \text{and} \qquad \psi'' = \pm\frac{P'}{\kappa\sqrt{nP\epsilon}}. \tag{43}$$

Hence

$$\eta = \frac{n}{2}\left(\frac{u}{u'}\frac{P'}{2P} + 1\right). \tag{44}$$

Finally, the slow-roll condition $|\eta| \ll 1$ then reads

$$\left|\frac{u}{u'}\frac{P'}{2P} + 1\right| \ll 1. \tag{45}$$

4. The acceleration condition

The slow-roll condition is useful for a non-phantom field because it is a necessary condition for inflating acceleration. However, in the case of a phantom field, the kinetic term is always negative and could take any large negative value; hence the slow-roll condition is not necessary for the acceleration condition. More generally, to ensure acceleration, equation (4) must be positive. It is straightforward to show that, obeying the acceleration condition $\ddot{a} > 0$, equation (5) takes the form

$$\epsilon\dot{\phi}(x)^2 < -\left(\frac{n-2}{2}\right)\frac{D}{a^n} + V. \tag{46}$$

With equations (8)–(10) and (17), the acceleration condition (46) in the NLS-type formulation is

$$E - P > -\frac{2}{n}\left(\frac{u'}{u}\right)^2 - \frac{nk}{2}\left(\frac{u^{2/n}}{u}\right)^2. \tag{47}$$

With the help of the non-linear Schrödinger-type equation (11), it is simplified to

$$u'' < \frac{2}{n}\frac{u'^2}{u}. \tag{48}$$

Using equations (31) and (32), the acceleration condition is just $\varepsilon < 1$ without using any slow-roll assumptions.

5. The WKB approximation

The WKB approximation can be assumed when the coefficient of the highest order derivative term in the Schrödinger equation is small or when the potential is very slowly varying. Equation (11) when $k = 0$ is linear. It is then

$$-\frac{1}{n}u'' + \left[\tilde{P}(x) - \tilde{E}\right]u = 0, \tag{49}$$

where $\tilde{P}(x) \equiv P(x)/n$ and $\tilde{E} \equiv E/n$. For a slowly varying $P(x)$ with the assumption $n \gg 1$, the solution of equation (49) can be written as $u(x) \simeq A\exp[\pm inW_0(x)]$, where $W_0(x) = W(x_0)$ is the lowest order term in the Taylor expansion of the function $W(x)$ in $(1/n)$ about $x = x_0$,

$$W(x) = W(x_0) + W'(x_0)\frac{(x - x_0)}{n} + \cdots. \tag{50}$$

Then an approximation

$$W(x) = \pm\frac{1}{n}\int_{x_1}^{x_2} k(x)\,\mathrm{d}x \simeq W_0(x) \tag{51}$$

is made by analogy to the method in time-independent quantum mechanics. The Schrödinger wavenumber is hence

$$k(x) = \frac{2\pi}{\lambda(x)} = \sqrt{n\left[\tilde{E} - \tilde{P}(x)\right]}, \tag{52}$$

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and the small variation in $\lambda(x)$ is

$$\frac{\delta\lambda}{\lambda(x)} = \left| \frac{\pi \tilde{P}'}{\sqrt{n}\left[\tilde{E} - \tilde{P}(x)\right]^{3/2}} \right| = \left| \frac{\pi P'}{[E - P(x)]^{3/2}} \right|. \tag{53}$$

For the WKB approximation, $\delta\lambda/\lambda(x) \ll 1$. In the real universe, we have $n = 3$ (dust) or $n = 4$ (radiation) which is not much greater than 1. However, if we are considering a range of very slowly varying potentials, $P' \simeq 0$ implying $\delta P/\delta x \sim 0$; hence $\delta k/\delta x \sim 0 \sim W'(x)$. Therefore $W(x) \simeq W_0(x)$ still holds in this range. Since $u(x) = a^{-n/2}$, using the WKB approximation, we get

$$a \sim A \exp\left[\pm(2/n)\mathrm{i} \int_{x_1}^{x_2} \sqrt{E - P(x)}\, \mathrm{d}x\right], \tag{54}$$

where A is a constant. Examples of Schrödinger potentials for exponential, power-law and phantom expansions are derived in [25]–[27]. These potentials are steep only in some small particular region but very slowly varying in most regions, especially at large values of $|x|$ for which the WKB approximation applies well.

6. The big rip singularity

When the field becomes a phantom, i.e. $\epsilon = -1$, in a flat FRLW universe it leads to a future big rip singularity [16, 17]. In a flat universe, when $w_{\rm eff} < -1$, i.e. for a phantom, the expansion obeys $a(t) \sim (t_{\rm a} - t)^q$, where $q = 2/3(1 + w_{\rm eff}) < 0$ is a constant over time and $t_{\rm a}$ is a finite time[2]. The NLS phantom expansion was studied in [27] with inclusion of the non-zero k case. Therein, the same expansion function is assumed, with constant $q < 0$, and x is related to the cosmic timescale, t, as $x(t) = (1/\beta)(t_{\rm a} - t)^{-\beta} + x_0$, so $u(x) = [\beta(x - x_0)]^\alpha$. Here $\alpha \equiv qn/(qn - 2)$ and $\beta \equiv (qn - 2)/2$, with the conditions $0 < \alpha < 1$ and $\beta < -1$ since $n > 0$ always. The first and second x derivatives of u are[3]

$$u'(x) = \alpha\beta[\beta(x - x_0)]^{\alpha-1}, \tag{55}$$

$$u''(x) = \alpha(\alpha - 1)\beta^2[\beta(x - x_0)]^{\alpha-2}, \tag{56}$$

where the exponents $\alpha - 1$ and $\alpha - 2$ are always negative. Using equations (20) and (22), then

$$\rho_{\rm tot} = \frac{12\alpha^2\beta^2}{\kappa^2 n^2}[\beta(x - x_0)]^{2(\alpha-1)} + \frac{3k}{\kappa^2}[\beta(x - x_0)]^{4\alpha/n}, \tag{57}$$

$$p_{\rm tot} = \frac{4\beta^2}{\kappa^2 n}[\beta(x - x_0)]^{2(\alpha-1)} \left[\left(1 - \frac{3}{n}\right)\alpha^2 - \alpha\right] - \frac{k}{\kappa^2}[\beta(x - x_0)]^{4\alpha/n} \tag{58}$$

$$= \frac{4u'^2}{\kappa^2 n}\left[\left(1 - \frac{3}{n}\right) - \frac{1}{\alpha}\right] - \frac{k}{\kappa^2}[\beta(x - x_0)]^{4\alpha/n}. \tag{59}$$

[2] The relation $q = 2/3(1 + w_{\rm eff}) < 0$ holds only when $k = 0$.

[3] Note that $(x - x_0)$ and β are negative hence $(x - x_0)^\alpha$, β^α, $(x - x_0)^{\alpha-1}$ and $\beta^{\alpha-1}$ are imaginary.

The big rip, $(a, \rho_{\rm tot}, |p_{\rm tot}|) \to \infty$, happens when $t \to t_{\rm a}^-$. In the NLS formulation, if $a \to \infty$, then $u \to 0^+$ (equation (8)). From above, we see that the conditions for the big rip singularity are

$$\begin{aligned} &t \to t_{\rm a}^- \Leftrightarrow x \to x_0^-, \\ &a \to \infty \Leftrightarrow u(x) \to 0^+, \\ &\rho_{\rm tot} \to \infty \Leftrightarrow u'(x) \to \infty, \\ &|p_{\rm tot}| \to \infty \Leftrightarrow u'(x) \to \infty. \end{aligned} \tag{60}$$

The effective equation of state $w_{\rm eff} = p_{\rm tot}/\rho_{\rm tot}$ can also be stated in NLS language as a function of x. Approaching the big rip, $x \to x_0^-$, and the effective equation of state approaches a value

$$\lim_{x \to x_0^-} w_{\rm eff} = \frac{n}{3}\left(1 - \frac{1}{\alpha}\right) - 1 = -1 + \frac{2}{3q}, \tag{61}$$

which is similar to the equation of state in the flat case.

7. Conclusions

We feature cosmological aspects of the NLS formulation of scalar field cosmology such as slow-roll conditions, the acceleration condition and the big rip. We conclude on these aspects in the standard Friedmann formulation before deriving them in the NLS formulation. We consider a non-flat FRLW universe filled with a scalar (phantom) field and barotropic fluid because, in the presence of a barotropic fluid density, the NLS-type formulation is consistent [26]. We obtain all NLS versions of the slow-roll parameters, the slow-roll conditions and the acceleration condition. This provides analytical tools in the NLS formulation. For a phantom field, due to its negative kinetic term, the slow-roll condition is not needed. When the NLS system is simplified to a linear equation (this happens when $k = 0$) as in time-independent quantum mechanics, we can apply the WKB approximation to the problem. When $n \gg 1$, the wavefunction is semi-classical, which is suitable for WKB approximation use. However, this does not work, since physically n cannot be much greater than unity, e.g. $n = 3$ for dust and $n = 4$ for radiation. However, the WKB approximation can still be clearly valid for a range of very slowly varying Schrödinger potentials $P(x)$ which were illustrated in [25]–[27]. Using the WKB approximation, we obtain an approximated scale factor function (equation (54)). For a flat universe with phantom expansion, the big rip singularity is its final fate. When the big rip happens, three quantities ($a(t), p(t)$ and $\rho(t)$) become infinite. Rewriting the singularity in NLS form (equation (60)), we can remove one infinity (see equation (60)). We found that near the big rip, $w_{\rm eff} \to -1 + 2/3q$ where $q < 0$ is a constant exponent of the expansion $a(t) \sim (t_{\rm a} - t)^q$. This limit is the same as the effective phantom equation of state in the case $k = 0$.

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