

Scalar field exact solutions for non-flat FLRW cosmology: a technique from non-linear Schrödinger-type formulation

Burin Gumjudpai

Received: 17 April 2008 / Accepted: 24 June 2008 / Published online: 10 July 2008
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Abstract We report a method of solving for canonical scalar field exact solution in a non-flat FLRW universe with barotropic fluid using non-linear Schrödinger (NLS)-type formulation in comparison to the method in the standard Friedmann framework. We consider phantom and non-phantom scalar field cases with exponential and power-law accelerating expansion. Analysis on effective equation of state to both cases of expansion is also performed. We speculate and comment on some advantage and disadvantage of using the NLS formulation in solving for the exact solution.

Keywords Scalar field cosmology · Non-linear Schrödinger equation · Power-law expansion · Exponential expansion

1 Introduction

In the past decade, it has been observed that universe is now in acceleration phase [1–9] while inflationary scenario of the early universe [10–15] is strongly confirmed by cosmic microwave background data [16–19]. In both circumstances, the universe experiences accelerating expansion which can be attained by exploiting some

This work is supported by a TRF-CHE Research Career Development Grant of the Thailand Research Fund and the Naresuan Faculty of Science Research Scheme.

B. Gumjudpai (✉)
Fundamental Physics and Cosmology Research Unit, The Tah Poe Academia Institute (TPTP),
Department of Physics, Naresuan University, Phitsanulok 65000, Thailand
e-mail: buring@nu.ac.th

B. Gumjudpai
School of Physics, Institute of Science, Suranaree University of Technology,
111 University Avenue, Nakhon Ratchasima 30000, Thailand

dynamical scalar field with time-dependent equation of state coefficient $w_\phi(t) < -1/3$, or a cosmological constant with $w_\Lambda = -1$ [20–22]. It was further suggested that the scalar field could have phantom behavior with $w_\phi < -1$ as one considers kinetic energy term in its Lagrangian density to be negative [23–28]. Strong supports to the phantom idea are from observations previously, e.g. combined cosmic microwave background, large scale structure survey and supernovae type Ia without assuming flat universe yields $w_{\phi,0} = -1.06^{+0.13}_{-0.08}$ [16] while using supernovae data alone assuming flat universe, $w_{\phi,0} = -1.07 \pm 0.09$ [29]. The subscript 0 denotes the value at present. Moreover, most recent WMAP five-year result [17, 18] combined with Baryon Acoustic Oscillation (BAO) of large scale structure survey from SDSS and 2dFGRS [30] and type Ia supernovae data from HST [7, 8], SNLS [9] and ESSENCE [29] assuming dynamical w with flat universe yields $-1.38 < w_{\phi,0} < -0.86$ at 95% confident level and $w_{\phi,0} = -1.12 \pm 0.13$ at 68% confident level [19]. With additional BBN constraint of limit of expansion rate [31, 32], $-1.32 < w_{\phi,0} < -0.86$ at 95% confident level and $w_{\phi,0} = -1.09 \pm 0.12$ at 68% confident level. This suggests that phantom field has firmed status in cosmology. However, phantom field does result in unwanted Big Rip singularity in a flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe [33–42]. This raises up many attempts to avoid the singularity based on both phenomenological and fundamental inspirations [43–50].

Recently, there are a few proposals for mathematical alternatives to the conventional Friedmann formulation of canonical scalar field cosmology, such as non-linear Ermakov–Pinney equation [51–56]. Moreover, a non-Ermakov–Pinney equation for the same system was also proposed in form of a non-linear Schrödinger-type equation (NLS) and it was found that solutions of the NLS-type equation are correspondent to solutions of the generalized Ermakov–Pinney equation [56, 57].¹ Conclusion of how to relate NLS quantities to quantities of standard Friedmann formulation is shown in [59] which gives extension to phantom field case. It also shows that the NLS wave function is in general non-normalizable. Expressing cosmological quantities in form of NLS quantities may suggest an alternative way of solving problems in scalar field cosmology. In such method, presumed knowledge of scale factor function with time $a(t)$ must be given first and later one can evaluate NLS potential based on $a(t)$ assumed. Here we attempt to solve for scalar field exact solution within the NLS framework in various cases with flat and non-flat spatial geometries. We compare the results to the solutions obtained in standard Friedmann formulation. Exponential expansion $a \sim \exp(t/\tau)$ and power-law expansion $a \sim t^q$ are assumed where τ are finite characteristic time and q is a positive constant.

This article is organized as follow. We express our cosmological system in Sect. 2 before introducing the NLS formulation in Sect. 3. Afterward, we consider each model of expansion separately. Exponential expansion is presented in Sect. 4 where we obtain exact solution from effective equation of state and later we solve for exact solution from Friedmann system. Afterward, in Sect. 5, we consider NLS formulation for the exponential expansion and solve for exact solutions in NLS framework. We analyze effective equation of state for the exponential expansion case in Sect. 6. When

¹ Considering Bianchi I scalar field cosmology, one can also construct a corresponding linear Schrödinger-type equation by redefining cosmological quantities [58].

considering power-law expansion, we work and organize the contents in the same spirit and order as in the previous sections. Beginning from Sects. 7, 8 and 9. Finally, conclusion is made in Sect. 10.

2 Cosmological system

To be realistic, two perfect fluids are considered in our system: barotropic fluid and scalar field fluid. The perfect barotropic fluid pressure p_γ and density ρ_γ obey an equation of state, $p_\gamma = (\gamma - 1)\rho_\gamma = w_\gamma\rho_\gamma$ while for scalar field, $p_\phi = w_\phi\rho_\phi$. Total density and total pressure are $\rho_{\text{tot}} = \rho_\gamma + \rho_\phi$ and $p_{\text{tot}} = p_\gamma + p_\phi$. The effective equation of state is weighed-value of these two components,

$$w_{\text{eff}} = \frac{\rho_\phi w_\phi + \rho_\gamma w_\gamma}{\rho_{\text{tot}}}. \tag{1}$$

For the barotropic fluid, its equation of state coefficient w_γ is written in term of n . We set $w_\gamma \equiv (n - 3)/3$ so that $n = 3(1 + w_\gamma) = 3\gamma$, hence $w_\gamma = -1$ corresponds to $n = 0$, $w_\gamma = -1/3$ to $n = 2$, $w_\gamma = 0$ to $n = 3$, $w_\gamma = 1/3$ to $n = 4$, and $w_\gamma = 1$ to $n = 6$. The conservation equation is therefore

$$\dot{\rho}_\gamma = -nH\rho_\gamma \tag{2}$$

with solution obtained directly,

$$\rho_\gamma = \frac{D}{a^n}, \tag{3}$$

therefore $p_\gamma = [(n - 3)/3](D/a^n)$, where a is scale factor, the dot denotes time derivative, $H = \dot{a}/a$ is Hubble parameter and $D \geq 0$ is a proportional constant. The scalar field considered here is minimally coupling to gravity with Lagrangian density, $\mathcal{L} = (1/2)\epsilon\dot{\phi}^2 - V(\phi)$, where $\epsilon = 1$ for non-phantom case and -1 for phantom case. Density and pressure of the field are given as

$$\rho_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi), \tag{4}$$

therefore

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\epsilon\dot{\phi}^2 - 2V(\phi)}{\epsilon\dot{\phi}^2 + 2V(\phi)}. \tag{5}$$

The field obeys conservation equation

$$\epsilon [\ddot{\phi} + 3H\dot{\phi}] + \frac{dV}{d\phi} = 0. \tag{6}$$

Considering FLRW universe, the Friedmann equation and acceleration equation are

$$H^2 = \frac{\kappa^2}{3}\rho_{\text{tot}} - \frac{k}{a^2}, \tag{7}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}\rho_{\text{tot}}(1 + 3w_{\text{eff}}), \tag{8}$$

where $\kappa^2 \equiv 8\pi G = 1/M_{\text{P}}^2$, G is Newton’s gravitational constant, M_{P} is reduced Planck mass, k is spatial curvature. Using Eqs. (3), (4), (6) and (7), it is straightforward to show that

$$\epsilon\dot{\phi}(t)^2 = -\frac{2}{\kappa^2} \left[\dot{H} - \frac{k}{a^2} \right] - \frac{nD}{3a^n}, \tag{9}$$

$$V(\phi) = \frac{3}{\kappa^2} \left[H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2} \right] + \left(\frac{n-6}{6} \right) \frac{D}{a^n}. \tag{10}$$

3 Non-linear Schrödinger-type formulation

Correspondence between NLS formulation for canonical scalar field cosmology with barotropic fluid was shown in [57] and was concluded recently in [59]. In the Schrödinger formulation, wave function $u(x)$ is related to scale factor in cosmology as

$$u(x) \equiv a(t)^{-n/2}, \tag{11}$$

and Schrödinger total energy E and Schrödinger potential $P(x)$ are linked to cosmology as

$$E \equiv -\frac{\kappa^2 n^2}{12} D, \tag{12}$$

$$P(x) \equiv \frac{\kappa^2 n}{4} a(t)^n \epsilon\dot{\phi}(t)^2. \tag{13}$$

These quantities satisfy NLS-type equation:

$$\frac{d^2}{dx^2} u(x) + [E - P(x)] u(x) = -\frac{nk}{2} u(x)^{(4-n)/n}. \tag{14}$$

The mapping from t to x is via $x = \sigma(t)$, such that

$$\dot{x}(t) = u(x), \tag{15}$$

$$\phi(t) = \psi(x). \tag{16}$$

We comment that the relation $\psi'(x)^2 = (4/n\kappa^2)P(x)$ in Ref. [57] does not include phantom field case. Modification is made in recent work [59] so that the solution

includes the phantom field case, therefore

$$\psi(x) = \pm \frac{2}{\kappa\sqrt{n}} \int \sqrt{\frac{P(x)}{\epsilon}} dx. \tag{17}$$

If $P(x) \neq 0$ and $n \neq 0$. There exists an inverse function of $\psi(x)$ as $\psi^{-1}(x)$. Therefore $x(t) = \psi^{-1} \circ \phi(t)$ and the scalar field potential, $V \circ \sigma^{-1}(x)$ can be expressed as function of time,

$$V(t) = \frac{12}{\kappa^2 n^2} \left(\frac{du}{dx}\right)^2 - \frac{2u^2}{\kappa^2 n} P(x) + \frac{12u^2}{\kappa^2 n^2} E + \frac{3ku^{4/n}}{\kappa^2}. \tag{18}$$

4 Exponential expansion

4.1 Solution solved from effective equation of state for $k = 0$ case

Exponential expansion reads

$$a(t) = \exp(t/\tau), \tag{19}$$

where τ is a positive constant. Flat universe undergoes exponential expansion only when $w_{\text{eff}} = -1$. The effective equation of state (Eq. 1), with Eqs. (4) and (5) can therefore be written as²

$$\epsilon \dot{\phi}^2 = -\frac{n}{3} \rho_\gamma, \tag{20}$$

which can be integrated directly, using Eq. (3), to

$$\phi(t) = \pm 2\tau \sqrt{\frac{D}{3n}} e^{-nt/2\tau} + \phi_0. \tag{21}$$

The solution above is obtained when assuming phantom scalar field, i.e. $\epsilon = -1$. If the scalar field is not phantom, the solution is imaginary.

4.2 Solution solved from Friedmann formulation

Another way to find the exact solution is to use Eq. (19), in Eq. (9). Therefore

$$\epsilon \dot{\phi}(t)^2 = \frac{2k}{\kappa^2} e^{-2t/\tau} - \frac{nD}{3} e^{-nt/\tau}, \tag{22}$$

² We are not considering a cosmological constant but a dynamical scalar field and a barotropic fluid which together yield $w_{\text{eff}} = -1$.

which gives an integration:

$$\phi(t) = \pm \int \sqrt{\frac{1}{\epsilon} \left(\frac{2k}{\kappa^2} e^{-2t/\tau} - \frac{nD}{3} e^{-nt/\tau} \right)} dt. \tag{23}$$

4.2.1 Simplest case

In the case of $k = 0$ and $D = 0$, the integration yields a constant ϕ_0 . Eq. (1) becomes $w_\phi = -1$. This is a cosmological constant as seen in simplest model of exponential expansion. When assuming only $k = 0$ and $\epsilon = -1$ but with $D \neq 0$, the solution of Eq. (23) is the same as the Eq. (21) previously. For a scalar field domination in a non-flat universe ($D = 0, k \neq 0$), the solution is

$$\phi(t) = \pm \frac{\tau}{\kappa} \sqrt{\frac{2k}{\epsilon}} e^{-t/\tau} + \phi_0. \tag{24}$$

where k and ϵ must have the same sign, otherwise the solution is imaginary.

4.2.2 The case of non-zero k and non-zero D

When k and D are both not negligible. Performing integration to the Eq. (23) is more complicated and could be impossible unless assumption of barotropic fluid type. When assuming a particular type of fluid in the integration, i.e. $n = 0, 2, 3, 4$ and 6 , analytical solution can be found for all n vales in complicated forms. For example, the simplest among these is dust case ($n = 3$) which has solution:

$$\phi(t) = \pm \frac{2\tau}{3D\sqrt{\epsilon}} \left(\frac{2k}{\kappa^2} - D e^{-t/\tau} \right)^{3/2} + \phi_0, \tag{25}$$

with additional rule that $k \geq 0$ and $\epsilon = 1$ otherwise it is imaginary. In the next section, we will show how to obtain solution in NLS formulation for dust and radiation cases.

5 Exponential expansion: solutions solved with NLS formulation

For exponential expansion, following Eqs. (11) and (15), we get

$$u(x) = \dot{x}(t) = \exp(-nt/2\tau). \tag{26}$$

Integrating the above equation, hence parameters x and t scale as

$$x(t) = -\frac{2\tau}{n} e^{-nt/2\tau} + x_0, \tag{27}$$

where x_0 is an integration constant. The reverse is

$$t(x) = -\frac{2\tau}{n} \ln [(-n/2\tau)(x - x_0)], \tag{28}$$

where the condition $x < x_0$ must be imposed. Now we can write wave function as

$$u(x) = -\frac{n}{2\tau}(x - x_0), \tag{29}$$

which is a linear function. Using Eq. (22), hence the Eq. (13) reads

$$P(t) = \frac{kn}{2} e^{(n-2)t/\tau} - \frac{\kappa^2 n^2 D}{12}. \tag{30}$$

Here the Schrödinger kinetic energy term is

$$T(t) = -\frac{kn}{2} e^{(n-2)t/\tau}. \tag{31}$$

Expressing in Schrödinger formulation, these functions are written in term of x ,

$$P(x) = \frac{kn}{2} \left[-\frac{n}{2\tau}(x - x_0) \right]^{-2(n-2)/n} - \frac{\kappa^2 n^2 D}{12}, \tag{32}$$

$$T(x) = -\frac{kn}{2} \left[-\frac{n}{2\tau}(x - x_0) \right]^{-2(n-2)/n}. \tag{33}$$

In order to obtain the scalar field potential $V(t)$, we use Eqs. (11), (12), (13) in Eq. (18), we finally obtain

$$V(t) = \frac{3}{\kappa^2 \tau^2} + \frac{2k}{\kappa^2} e^{-2t/\tau} + \left(\frac{n-6}{6} \right) D e^{-nt/\tau}. \tag{34}$$

which is checked by using Eq. (19) in standard formula (10). We use Eq. (32) in Eq. (17), then

$$\psi(x) = \frac{\pm 2}{\kappa \sqrt{n}} \times \int \sqrt{\frac{kn}{2\epsilon} \left[-\frac{n}{2\tau}(x - x_0) \right]^{-2(n-2)/n} - \frac{\kappa^2 n^2 D}{12\epsilon}} dx. \tag{35}$$

We will integrate this equation in cases of $k = 0$ and $k \neq 0$.

5.1 The case $k = 0$

When $k = 0$ and $D \neq 0$ integrating Eq. (35) and transforming x to t with Eq. (27) yields same result as Eq. (21) obtained by solving effective equation of state equation or by integrating from the Friedmann formulation. Real solution exists only when the

scalar field is phantom. With the solution (21), the scalar field potential in term of ϕ , reads

$$V(\phi) = \frac{3}{\kappa^2 \tau^2} + \left(\frac{n-6}{6}\right) \frac{3n}{4\tau^2} (\phi - \phi_0)^2. \tag{36}$$

5.2 The case $k \neq 0$

When $k \neq 0$ and $D \neq 0$, the integral (35) can be integrated yielding complicated hypergeometric function even when n is not specified. The case $n = 0$ is excluded from our consideration by the reason mentioned in Sect. 3. For naturalness, we consider radiation ($n = 4$) and dust ($n = 3$).

5.2.1 Radiation case

Radiation fluid corresponds to $n = 4$, the integral (35) becomes

$$\psi(x) = \pm \frac{1}{\kappa} \int \sqrt{-\frac{k\tau}{\epsilon} \frac{1}{(x-x_0)} - \frac{4}{3} \frac{\kappa^2 D}{\epsilon}} dx. \tag{37}$$

Here x could be negative, ϵ can possibly be either ± 1 . The solution in radiation case is

$$\begin{aligned} \psi(x) = & \pm \sqrt{\frac{1}{\epsilon} \left[-\frac{4}{3} D(x-x_0)^2 - \frac{k\tau}{\kappa^2} (x-x_0) \right]} \pm \frac{k\tau}{4\kappa^2} \sqrt{\frac{3}{D\epsilon}} \\ & \times \arctan \left\{ \frac{[8\kappa^2 D(x-x_0)/3\epsilon] + k\tau/\epsilon}{[4\kappa\sqrt{D}/(\epsilon\sqrt{3})]\sqrt{-[4\kappa^2 D(x-x_0)^2/3] - k\tau(x-x_0)}} \right\} + \psi_0, \end{aligned} \tag{38}$$

allowing only $\epsilon = 1$ case for the solution to be real. Transforming x scale to the t scale using Eq. (27), the solution therefore reads

$$\begin{aligned} \phi(t) = & \pm \sqrt{\frac{1}{\epsilon} \left(-\frac{D\tau^2}{3} e^{-4t/\tau} + \frac{k\tau^2}{2\kappa^2} e^{-2t/\tau} \right)} \pm \frac{k\tau}{4\kappa^2} \sqrt{\frac{3}{D\epsilon}} \\ & \times \arctan \left\{ \frac{-[4\kappa^2 D\tau/(3\epsilon)] e^{-2t/\tau} + k\tau/\epsilon}{[4\kappa\sqrt{D}/(\epsilon\sqrt{3})]\sqrt{-(\kappa^2 D\tau^2/3) e^{-4t/\tau} + (k\tau^2/2) e^{-2t/\tau}}} \right\} + \phi_0. \end{aligned} \tag{39}$$

The solution above, when assuming $k = 0$, reduces to the solution (21) when $n = 4$, confirming the correctness of the result obtained.

5.2.2 Dust case

The integral (35) in the dust case $n = 3$ reads

$$\psi(x) = \pm \frac{2}{\kappa\sqrt{3}} \int \sqrt{\left(\frac{3}{2}\tau^2\right)^{1/3} \frac{k}{\epsilon} \frac{1}{(x-x_0)^{2/3}} - \frac{3}{4} \frac{\kappa^2 D}{\epsilon}} dx, \tag{40}$$

with solutions

$$\psi(x) = \pm \sqrt{\frac{D}{\epsilon}} \left[\left(\frac{3\tau^2}{2}\right)^{1/3} \frac{4k}{3\kappa^2 D} - (x-x_0)^{2/3} \right]^{3/2} + \psi_0. \tag{41}$$

With similar procedure to the radiation case, using (27), the solution is therefore,

$$\phi(t) = \pm \frac{2\tau}{3D\sqrt{\epsilon}} \left(\frac{2k}{\kappa^2} - De^{-t/\tau} \right)^{3/2} + \phi_0, \tag{42}$$

which is the same as Eq. (25) derived from Friedmann formulation. This solution when assuming $k = 0$ is exactly the same as the solution (21) when $n = 3$ (dust fluid). This also confirms that our results from NLS formulation are correct. The NLS solution can solve the case when k and D are non-zero together without knowing n value while the standard procedures in Sect. 4.2 cannot unless assuming a particular value $n = 0, 2, 3, 4, 6$. However, it must be noticed that one cannot reduce the NLS solutions (39) and (42) to the $D = 0$ case directly since there are mixed multiplication term of n and k in the solution and also the value of n has already been put in. Hence setting $D = 0$ in (39) and (42) cannot be considered as a pure scalar field dominant case.

6 Exponential expansion: analysis on effective equation of state coefficient

The exponential expansion in our scenario is caused from mixed effect of fluids and spatial curvature. We discuss mixed effect on equation of state here. Definition of effective equation of state coefficient, $w_{\text{eff}} = (\rho_\phi w_\phi + \rho_\gamma w_\gamma) / \rho_{\text{tot}}$ together with Eqs. (4), (22) and (34) in context of exponential expansion becomes

$$w_{\text{eff}} = \frac{-1 - (k\tau^2/3)e^{-2t/\tau}}{1 + k\tau^2 e^{-2t/\tau}}, \tag{43}$$

which is infinite when

$$t = \frac{\tau}{2} \ln(-k\tau^2). \tag{44}$$

Infinity can possibly happen only when $k = -1$ because logarithm function forbids negative domain. In order to acquire exponential expansion in flat universe, one needs

to have $w_{\text{eff}} = -1$, but this is not true when k term is non-trivial. Therefore we can only express w_ϕ in term of w_{eff} as

$$w_\phi = \frac{[(3k/\kappa^2)e^{-2t/\tau} + 3/(\kappa^2\tau^2)]w_{\text{eff}} - [(n-3)/3]De^{-nt/\tau}}{(3k/\kappa^2)e^{-2t/\tau} + 3/(\kappa^2\tau^2) - De^{-nt/\tau}}, \tag{45}$$

for exponential expansion. The Eq. (43) does not depend on properties (n) or amount (D) of the barotropic fluid. It reduces to $w_{\text{eff}} = -1$ when $k = 0$ as expected. Considering Eq. (45), if $D = 0$ and $k = 0$, it yields $w_\phi = w_{\text{eff}}$ while setting $D = 0$ alone also gives the same result.

7 Power-law expansion

7.1 Bound value of $\phi(t)$ from effective equation of state for $k = 0$ case

In power-law expanding universe, scale factor evolves with time as

$$a(t) = t^q, \tag{46}$$

where $q > 0$ is a constant. In flat ($k = 0$) universe, it is known that the power-law expansion, is attained when $-1 < w_{\text{eff}} < -1/3$ where $q = 2/[3(1 + w_{\text{eff}})]$. The effective equation of state (Eq. 1), hence is a condition

$$-\frac{n}{3}\rho_\gamma < \epsilon\dot{\phi}^2 < \frac{2}{3}\left(\frac{\epsilon\dot{\phi}^2}{2} + V\right) + \left(\frac{2-n}{3}\right)\rho_\gamma, \tag{47}$$

i.e. $0 < \epsilon\dot{\phi}^2 + \rho_\gamma n/3 < (2/3)\rho_{\text{tot}}$. Both values of ϵ can be assigned and the power-law expansion is sustained as long as the condition is satisfied.

7.2 Solution solved from Friedmann formulation

If we directly consider Eq. (9), the solution for power-law expansion is an integration:

$$\phi(t) = \pm \int \sqrt{\frac{1}{\epsilon} \left(\frac{2q}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} - \frac{nD}{3t^{qn}} \right)} dt. \tag{48}$$

7.2.1 Simplest case

Simplest integration case is when $k = 0$ and $D = 0$. The solution of Eq. (48) is well known [60],

$$\phi(t) = \pm \sqrt{\frac{2q}{\epsilon\kappa^2}} \ln t + \phi_0, \tag{49}$$

provided that q and ϵ have the same sign. Considering power-law inflation, the WMAP five-year combined analysis based on flat and scalar field domination assumption yields $q > 60$ at more than 99 % of confident level otherwise excluded while $q \sim 120$ is at boundary of 68% confident level [19]. These results base on single field model which we can applied the above solution to. When assuming only $k = 0$ with $D \neq 0$, the solution of Eq. (48) is

$$\phi(t) = \pm \frac{1}{qn - 2} \sqrt{\frac{2q}{\epsilon \kappa^2}} \left\{ \ln \left[\frac{t^{-qn+2}}{\left(1 + \sqrt{1 - (nD\kappa^2/6q)t^{-qn+2}}\right)^2} \right] + 2\sqrt{1 - \left(\frac{nD\kappa^2}{6q}\right)t^{-qn+2}} + \ln \left(-\frac{nD\kappa^2}{6q}\right) \right\} + \phi_0, \tag{50}$$

where, when $q = 2/n$, the field has infinite value. The last logarithmic term in the bracket is an integrating constant which is valid only when $q < 0$. To attain power-law expansion, q must be positive. Hence, this term is not defined for power-law expansion. We will see later that the NLS result does not have this problem. For the reverse case, $D = 0, k \neq 0$, the solution is

$$\phi(t) = \pm \frac{1}{q - 1} \sqrt{\frac{2q}{\epsilon \kappa^2}} \times \left\{ \ln \left[\frac{t^{q-1}}{\sqrt{k/q}} \left(1 + \sqrt{\left(\frac{k}{q}\right)t^{-2q+2} + 1}\right) \right] - \sqrt{\left(\frac{k}{q}\right)t^{-2q+2} + 1} \right\} + \phi_0, \tag{51}$$

which becomes infinite when $q = 1$. The values of q, k and ϵ must have the same sign in all terms of the solution otherwise becoming imaginary. Hence, for $q > 0$, the condition for the solution to be valid is $k = 1$ and $\epsilon = 1$.

7.2.2 The case of non-zero k and non-zero D

When considering non-negligible value of both k and D , the Eq. (48) cannot be integrated analytically except when setting $n = 2$ ($w_\gamma = -1/3$) which is not natural fluid. Hence it is not considered.

8 Power-law expansion: solutions solved with NLS formulation

Power-law expansion cosmology in NLS-type formulation is presented and concluded in [59]. Important functions needed for evaluating the field exact solutions are

$$x = \sigma(t) = -\frac{t^{-\beta}}{\beta} + x_0, \tag{52}$$

$$t(x) = \frac{1}{[-\beta(x - x_0)]^{1/\beta}}, \tag{53}$$

$$\epsilon \dot{\phi}(t)^2 = \frac{2q}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} - \frac{nD}{3t^{qn}}, \tag{54}$$

$$P(x) = \frac{2qn}{(qn - 2)^2} \frac{1}{(x - x_0)^2} + \frac{kn}{2} \left[\frac{-2}{(qn - 2)(x - x_0)} \right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12}, \tag{55}$$

$$V(t) = \frac{q(3q - 1)}{\kappa^2 t^2} + \frac{2k}{\kappa^2 t^{2q}} + \left(\frac{n - 6}{6} \right) \frac{D}{t^{qn}}. \tag{56}$$

We use Eq. (55) in Eq. (17), then

$$\psi(x) = \frac{\pm 2}{\kappa \sqrt{n}} \times \int \sqrt{\frac{2qn}{\epsilon(qn-2)^2} \frac{1}{(x-x_0)^2} + \frac{kn}{2\epsilon} \left[\frac{-2}{(qn-2)(x-x_0)} \right]^{2q(n-2)/(qn-2)} - \frac{\kappa^2 n^2 D}{12\epsilon}} dx. \tag{57}$$

We consider the solution in cases of $k = 0$ and $k \neq 0$. Recall that setting $D = 0$ cannot be considered as an absence of barotropic fluid due to existence of n in the other terms.

8.1 The case $k = 0$

Solution to the integral (57) for $k = 0$ case is

$$\psi(x) = \pm \sqrt{\frac{8q}{\epsilon \kappa^2 (qn - 2)^2}} \left\{ -\sqrt{1 - \left[\frac{\kappa^2 D n (qn - 2)^2}{24q} (x - x_0)^2 \right]} + \ln \left[\frac{1 + \sqrt{1 - \left[\frac{\kappa^2 D n (qn - 2)^2}{24q} (x - x_0)^2 \right]}}{(x - x_0)} \frac{4qn}{\epsilon (qn - 2)^2} \right] \right\}. \tag{58}$$

Transforming to the t variable using Eq. (52), we obtain,

$$\phi(t) = \pm \frac{1}{qn - 2} \sqrt{\frac{2q}{\epsilon \kappa^2}} \left\{ \ln \left[\frac{t^{-qn+2}}{\left(1 + \sqrt{1 - \left(\frac{nD\kappa^2}{6q} \right) t^{-qn+2}} \right)^2} \right] + 2\sqrt{1 - \left(\frac{nD\kappa^2}{6q} \right) t^{-qn+2}} + \ln \left(\frac{qn - 2}{2qn} \right)^2 \right\} + \phi_0. \tag{59}$$

This solution differs from the solution (50) only the last logarithmic term in the bracket which is only an integrating constant. When $q = 2/n$ or $n = 0$, the field has infinite value. The last logarithmic term does not restrict the sign of q . Only q and ϵ must have the same sign for the solution to be real.

8.2 The case $k \neq 0$

In case of non-zero k and non-zero D , the integral (57) cannot be integrated analytically even when assuming each n value except when $n = 2$ which is not natural fluid.

9 Power-law expansion: analysis on effective equation of state coefficient

Similar to the analysis in Sect. 6, mixed effect of the two fluids and spatial curvature results in power-law expansion. The coefficient w_{eff} , with Eqs. (4), (54) and (56), reads

$$w_{\text{eff}} = \frac{(-3q^2 + 2q)t^{2q-2} - k}{3q^2 t^{2q-2} + 3k}, \tag{60}$$

which becomes infinity if

$$t = \left(\frac{-k}{q^2}\right)^{1/(2q-2)}. \tag{61}$$

We can also express w_ϕ in term of w_{eff} as

$$w_\phi = \frac{[(3q^2/\kappa^2)t^{-2} + (3k/\kappa^2)t^{-2q}]w_{\text{eff}} - [(n-3)/3]Dt^{-qn}}{(3q^2/\kappa^2)t^{-2} + (3k/\kappa^2)t^{-2q} - Dt^{-qn}}, \tag{62}$$

for power-law expansion. The Eq. (62), when $D = 0$ and $k = 0$, yields $w_\phi = w_{\text{eff}}$ as expected. Similar to the case of exponential expansion, setting $D = 0$ alone also yields $w_\phi = w_{\text{eff}}$. In flat universe, power-law expansion happens when w_{eff} lies in an interval $(-1, -1/3)$. But in $k \neq 0$ universe, it is no longer true. Considering flat universe, setting $k = 0$ in Eq. (60) yields $q = 2/[3(1 + w_{\text{eff}})]$. The condition $-1 < w_{\text{eff}} < -1/3$ therefore corresponds to $q > 0$ as known. The condition also yields

$$-1 - (1 + w_\gamma)\frac{\rho_\gamma}{\rho_\phi} < w_\phi < -\frac{1}{3} - \left(\frac{1}{3} + w_\gamma\right)\frac{\rho_\gamma}{\rho_\phi}. \tag{63}$$

If there is more non-negligible radiation fluid (with $w_\gamma = 1/3$), it is noticed that the interval shifts to the more left. For example, setting $\rho_\gamma = 0.1\rho_\phi$, the interval shifts to about $-1.133 < w_\phi < -0.4$. If we assume more realistic situation when dust (dark matter and other matter elements) is presented. The dust density and dark energy is about 28% and 72% of total density, therefore $\rho_\gamma \simeq (28/72)\rho_\phi \simeq 0.389\rho_\phi$, the

interval is $-1.389 < w_\phi < -0.463$ which covers valid range of recent observational data, assuming dynamical w with flat universe, $-1.38 < w_{\phi,0} < -0.86$ at 95% confident level [19].

10 Conclusions

This letter reports and demonstrates a method of solving for canonical scalar field exact solution in a non-flat FLRW universe with barotropic fluid using NLS-type formulation in comparison with the method in the standard Friedmann framework. We consider phantom and non-phantom scalar field cases with exponential and power-law accelerating expansion. We evaluate all NLS quantities needed to find the solution, e.g. non-normalizable wave function and Schrödinger potential. Our process is reverse to a problem solving in quantum mechanics that the wave function is expressed first by the expansion function, $a(t)$ before evaluating the Schrödinger potential based on a known expansion function. In NLS formulation the total energy E is negative. We do an analysis on effective equation of state to both cases of expansion. We express w_{eff} in term of q and k . In a flat universe, in order to have power-law expansion, the interval $(-1, -1/3)$ of the w_ϕ , is shifted leftward to more negative if more barotropic fluid density is presented.

Within framework of the standard Friedmann formulation, we obtained exact solution in various cases. Later we solved the problem using NLS formulation, in which the wave function is equivalent to the scalar field exact solution. NLS method is restricted by the fact that its scalar field solution is valid only when the barotropic fluid density is presented. Setting $D = 0$ does not imply the absence of barotropic fluid because the barotropic fluid parameter n still appears in the other terms of the Schrödinger potential. Therefore NLS formulation cannot be applied to situation when the scalar field is dominant and $D \sim 0$. Hence it is more suitable for a system of dark energy and dust dark matter fluid. This is a disadvantage point of the NLS formulation. Transforming from standard Friedmann formulation to NLS formulation makes n appear in all terms of the integrand and also changes fluid density term D from time-dependent term to a constant E . Hence the number of x (or equivalently t)-dependent terms is reduced by one. This is a good aspect of the NLS. In both Friedmann formulation and NLS formulation, the solutions when $k \neq 0$ and $D \neq 0$ are difficult or might be impossible to solve unless assuming values of q and n . Hence reduction number of x -dependent term helps simplifying the integration. There are also other good aspects of NLS formulation. Firstly, in the case of exponential expansion with NLS formulation, the solution when $k \neq 0$ and $D \neq 0$ can be obtained without assuming n value while $n = 0, 2, 3, 4, 6$ must be given if working within Friedmann formulation. Secondly, for power-law expansion with $k = 0$, the result (59) obtained from NLS formulation has integrating constant that does not restrict q value while (50) obtained from Friedmann formulation needs $q < 0$ which violates power-law expansion condition ($q > 0$). For power-law expansion, the most difficult case is when $k \neq 0$ with $D \neq 0$. In both formulations, the integral cannot be integrated unless assuming $n = 2$ (equivalent to $w_\gamma = -1/3$) which is not a physical fluid. We introduce here alternative method to obtain scalar field exact solution with advantage over and disadvantage to

standard Friedmann formulation. The NLS formulation could render more interesting techniques for scalar field cosmology.

Acknowledgments B. G. is a TRF Research Scholar of the Thailand Research Fund. B. G. gives a special thank for hospitalities, supports during his research visits and for seminal invitations, to Yupeng Yan, Prapan Manyum and Chinorat Kobdaj at Suranaree University of Technology, to Department of Physics of Ubon Rajathaneey University and to the organizers of the Forth Aegean Summer School on Black Holes & the First Annual School of the EU Network *UniverseNet*-The Origin of the Universe in Mytilini, Island of Lesbos, Greece where this work was partially completed.

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