Path Integral Approach to Single-Particle Motion in Forced Harmonic Potential

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May the whole of my life dedicated to
my father
my dream
and my country

Pimpimon Khumwong, 14 February 2006

To my father and mother, lords of my life.

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The candidate has passed oral examination by members of examination panel. This report has been accepted by the panel as partially fulfilment of the course 261493: Independent Study.

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Abstract

In this work, we study the single-particle motion in the forced harmonic potential by Feynman path integral method. The propagator is evaluated and analyzed. We compare this result with the wave mechanical solution in case of the constant external force. Our new method gives the same ground state eigenenergy and wave function as the technique of the wave mechanics. This shows that the path integral method corresponds to the wave mechanical technique.

Contents

Chapter 1

Introduction

Mechanics explains motion of particles, it can be roughly divided into

1. **Classical mechanics** describes the particle motion in macroscopic scale based on Newton's three laws of motion.
2. **Quantum mechanics** describes the particle motion in microscopic scale based on Heisenberg's uncertainty principle.

In case of quantum mechanics, it can be classified into two different forms

2.1 **Schrödinger's wave mechanics**

The quantum state of the system of interest is represented by wave function which can be solved from Schrödinger equation.

2.2 **Heisenberg's matrix mechanics**

The dynamical variables of the system are written in the from of matrix.

Both theories are based on Hamiltonian mechanics. After that, R.P. Feynman proposed the third formulation of quantum mechanics based on Lagrangian mechanics. He developed Dirac's idea which suggested that transition amplitude $< x_2, t_2|x_1, t_1 >$ corresponds to $\exp\left(\frac{\mathrm{i}}{\hbar}\int_{t_1}^{t_2} L\mathrm{d}t\right)$. This method is called *Path Integrals.*

1.1 Background

Path integral is the alternative method to describe the behavior of quantum system. It is the most powerful in many fields of physics. Due to this reason, it is interesting to apply this method to some simple systems and hoped that it can be developed in the future works.

1.2 Objectives

1.2.1 To study the single-particle motion in the forced harmonic potential by path integral method.
1.2.2 To calculate and analyze the single-particle propagator in the forced harmonic potential.

1.3 Framework

This work focuses on the propagator of the single-particle moving in the forced harmonic potential calculated by path integral method.

1.4 Expected Use

1.4.1 To attain knowledge and understanding in path integral.
1.4.2 To be able to calculate single-particle propagator in forced harmonic potential.
1.4.3 To apply the path integral method to other systems in the future work.

1.5 Tools

- Computer
- Textbooks on physics and mathematics

1.6 Procedure

1.6.1 Study path integral.
1.6.2 Study Green's function.
1.6.3 Calculate single-particle propagator in forced harmonic potential.
1.6.4 Calculate wave function from propagator.
1.6.5 Make conclusion and comment.

1.7 Outcomes

We finally found that the ground-state wave functions and energies are the same either evaluating by Path Integral method or by using Schrödinger's equation in conventional quantum mechanics.

Chapter 2

Introduction to path integrals

Consider the motion of the single particle in one dimension from (x_a, t_a) to (x_b, t_b). In macroscopic scale the motion is described by classical mechanics which states that the particle shall move along the path with minimum action as shown in Fig. 2.1. The action is defined as

$$S[b, a] = \int_{t_a}^{t_b} L(x, \dot{x}, t)\mathrm{d}t, \tag{2.1}$$

where

S is the action of the particle,

$L(x, \dot{x}, t)$ is the Lagrangian of the system.

The particle moves in the way that minimizes the action S. This leads to the Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0. \tag{2.2}$$

The motion can be solved from equation (2.2).

The situation is completely different in quantum mechanics. There are many possible paths from (x_a, t_a) to (x_b, t_b) as shown in Fig. 4.1.

The total amplitude of the particle $K(b, a)$ to go from a to b can be found from the sum of contribution of the amplitude $\phi[x(t)]$ from each path.

$$K(b, a) = \sum_{\text{all possible path}} \phi[x(t)]. \tag{2.3}$$

The probability $P(b, a)$ to go from the point x_a at time t_a to the point x_b at t_b is the absolute square of the total amplitude

$$P(b, a) = \left|K(b, a)\right|^2. \tag{2.4}$$

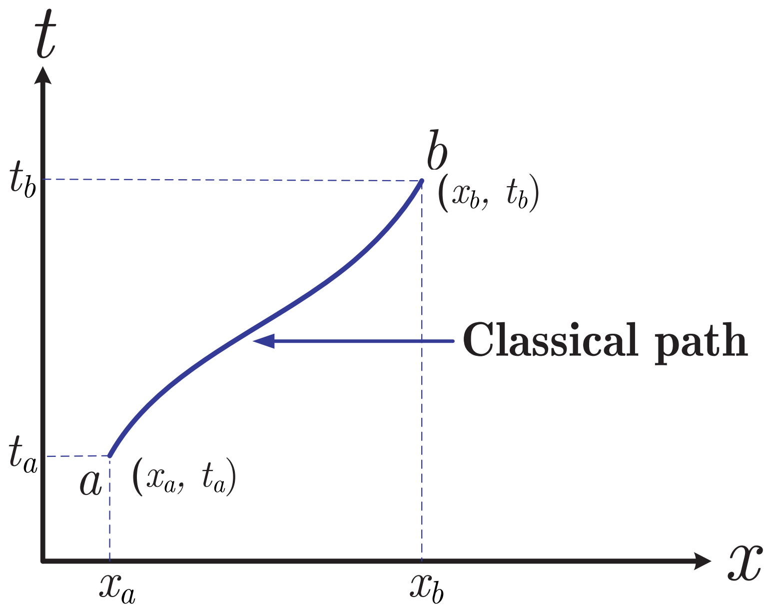

Figure 2.1: The single-particle path of motion according to classical mechanics from (x_a, t_a) to (x_b, t_b).

Feynman conjectured that the contribution of a path has phase proportional to the action S associated with each path

$$\phi[x(t)] = (\text{Const.})\exp\Big\{(i/\hbar)S[x(t)]\Big\}. \tag{2.5}$$

We shall write the sum over all paths in a less restrictive notation as

$$K(b,a) = \int_a^b \exp\Big\{(i/\hbar)S[b,a]\Big\}\mathrm{D}x(t) \tag{2.6}$$

which we shall call a path integral and $K(b,a)$ is sometimes called the propagator.

2.1 Events occurring in succession

From Fig. 2.3, the action from a to b can be separated into two parts as

$$S[b,a] = S[b,c] + S[c,a],$$

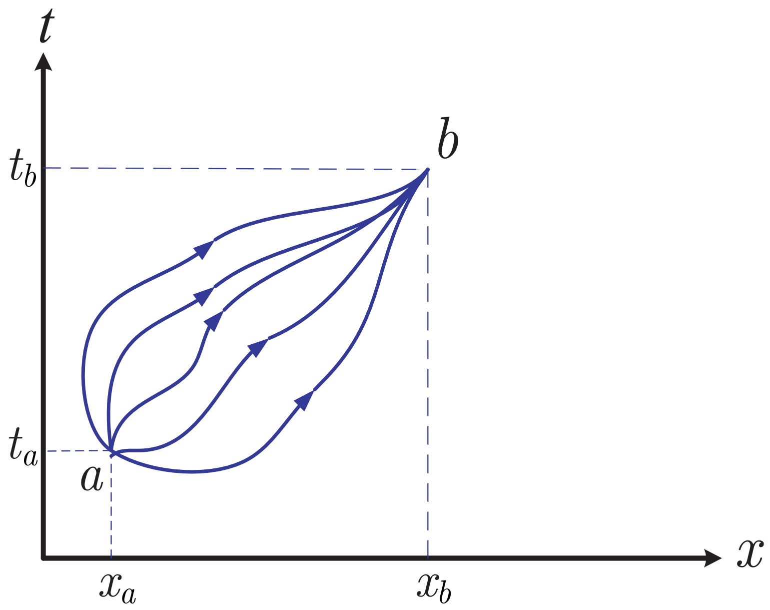

Figure 2.2: The possible paths of the quantum mechanics from (x_a, t_a) to (x_b, t_b).

then the propagator becomes

$$
\begin{aligned}
K(b,a) &= \int_a^b \exp\Big\{(i/\hbar)S[b,c]\Big\}\exp\Big\{(i/\hbar)S[c,a]\Big\}\mathrm{D}x(t) \\
&= \int_{x_c}\int_c^b \exp\Big\{(i/\hbar)S[b,c]\Big\}\Big(\int_a^c \exp\Big\{(i/\hbar)S[c,a]\Big\}\mathrm{D}x(t)_{a\to c}\Big)\mathrm{d}x_c\mathrm{D}x(t)_{c\to b} \\
&= \int_{x_c}\int_c^b \exp\Big\{(i/\hbar)S[b,c]\Big\}K(c,a)\mathrm{D}x(t)_{c\to b}\mathrm{d}x_c \\
&= \int_{x_c}\int_c^b \exp\Big\{(i/\hbar)S[b,c]\Big\}\mathrm{D}x(t)_{c\to b}K(c,a)\mathrm{d}x_c \\
K(b,a) &= \int_{x_c} K(b,c)K(c,a)\mathrm{d}x_c. \qquad (2.7)
\end{aligned}
$$

The propagator for a particle going from a to b can be computed from the rules:

1. The propagator of motion from a to b is the sum over all possible values of x_c of amplitudes for the particle of motion from a to c and then from c to b.

2. The amplitude of motion from a to c and then to b is the propagator of motion from a to c times the propagator of motion from c to b.

Thus we have the rule: *Multiplying Amplitudes for events occurring in succession in time gives the total amplitude.*

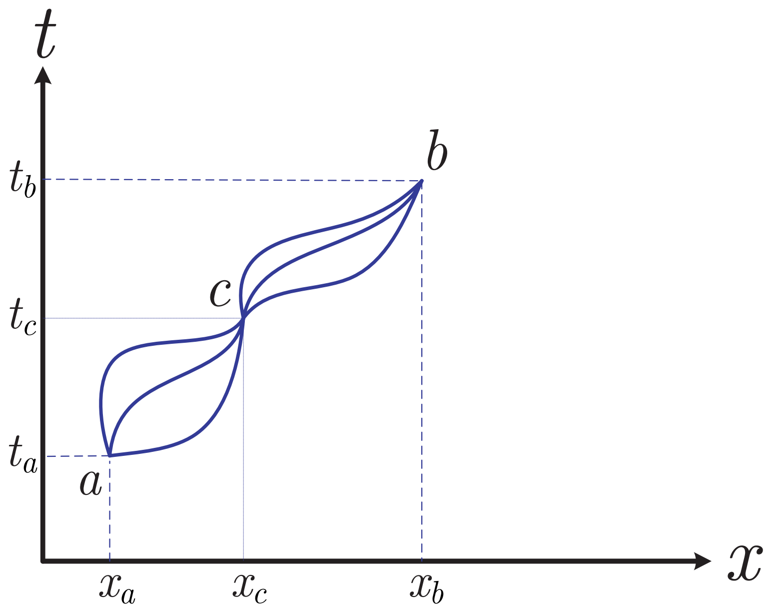

Figure 2.3: The paths of motion of the particle from (x_a, t_a) to (x_b, t_b), where each path passed (x_c, t_c)

We can continue this process until we have the time scale divided into N intervals. The result is

$$K(b,a) = \int_{x_1}\int_{x_2}...\int_{x_{N-1}} K(b,N-1)K(N-1,N-2)...K(i+1,i)...K(1,a)\mathrm{d}x_1\mathrm{d}x_2...\mathrm{d}x_{N-1} \tag{2.8}$$

2.2 The wave function

From Fig. 2.1 where

$K(x_b, t_b; x_a, t_a)$ is the probability amplitude of finding the particle at (x_b, t_b) starting from (x_a, t_a).

$\left|K(x_b, t_b; x_a, t_a)\right|^2$ is the probability of finding the particle at (x_b, t_b).

$\psi(x_b, t_b)$ is the probability amplitude of finding the particle at (x_b, t_b) (we are not interested in the origin of the particle's motion).

$\left|\psi(x_b, t_b)\right|^2$ is the probability of finding the particle at (x_b, t_b).

So we can conclude that

$$K(x_b, t_b; x_a, t_a) = \psi(x_b, t_b).$$

From

$$K(x_b,t_b;x_a,t_a) = \int_{-\infty}^{\infty} K(x_b,t_b;x_c,t_c)K(x_c,t_c;x_a,t_a)\mathrm{d}x_c. \tag{2.9}$$

We can rewrite equation (2.9) as

$$\psi(x_b,t_b) = \int_{-\infty}^{\infty} K(x_b,t_b;x_c,t_c)\psi(x_c,t_c)\mathrm{d}x_c. \tag{2.10}$$

From equation (2.10), we can see that the wave function $\psi(x,t)$ can be solved when we know the propagator of the system.

In standard quantum mechanics, at time t_c the state of system is $\big|\psi(t_c)\big\rangle$. Then state at a new time t_b is

$$\big|\psi(t_b)\big\rangle = \hat{u}(t_b,t_c)\big|\psi(t_c)\big\rangle,$$

where $\hat{u}(t_b,t_c)$ is time evolution operator of the quantum state from time t_c to t_b.

The wave function is generated by performing the inner product of $\big|\psi(t_b)\big\rangle$with $\big\langle x_b\big|$, we get

$$\begin{aligned}\psi(x_b,t_b) &= \big\langle x_b\big|\psi(t_b)\big\rangle \\ &= \big\langle x_b\big|\hat{u}(t_b,t_c)\big|\psi(t_c)\big\rangle.\end{aligned}$$

From the completeness relation

$$\int_{-\infty}^{\infty} \mathrm{d}x\big|x\big\rangle\big\langle x\big| = \hat{1}.$$

The wave function can be written as

$$\begin{aligned}\psi(x_b,t_b) &= \big\langle x_b\big|\hat{u}(t_b,t_c)\int_{-\infty}^{\infty}\mathrm{d}x_c\big|x_c\big\rangle\big\langle x_c\big|\psi(t_c)\big\rangle \\ &= \int_{-\infty}^{\infty}\big\langle x_b\big|\hat{u}(t_b,t_c)\big|x_c\big\rangle\big\langle x_c\big|\psi(t_c)\big\rangle\mathrm{d}x_c \\ \psi(x_b,t_b) &= \int_{-\infty}^{\infty}\big\langle x_b\big|\hat{u}(t_b,t_c)\big|x_c\big\rangle\psi(x_c,t_c)\mathrm{d}x_c. \qquad (2.11)\end{aligned}$$

where

$\big\langle x_b\big|\hat{u}(t_b,t_c)\big|x_c\big\rangle$ is *transition amplitude* from state $\big|x_c\big\rangle$ to state $\big|x_b\big\rangle$.

By comparing equation (2.10) with equation (2.11), finally we get

$$\big\langle x_b\big|\hat{u}(t_b,t_c)\big|x_c\big\rangle = K(x_b,t_b;x_c,t_c), \tag{2.12}$$

2.3 The propagator of the quadratic Lagrangian

The simplest path integrals are those of which all of the variables appear up to the second degree in exponent. We shall call them Gaussian integrals. In quantum mechanics this corresponds to a case in which the action S involves the path $x(t)$ up to and including the second power.

To illustrate how the method works in such a case, consider a particle whose Lagrangian appear in quadratic form

$$L = a(t)\dot{x}^2 + b(t)\dot{x}x + c(t)x^2 + d(t)\dot{x} + e(t)x + f(t). \tag{2.13}$$

In classical mechanics, form of the action integral is

$$S[x(t)] = \int_{t_1}^{t_2} L(x, \dot{x}, t)\mathrm{d}t. \tag{2.14}$$

Therefore the particle's action of the Lagrangian of equation (2.13) of particle moving from $(x_{\mathrm{a}}, t_{\mathrm{a}})$ to $(x_{\mathrm{b}}, t_{\mathrm{b}})$ is written as

$$S[x(t)] = \int_{t_{\mathrm{a}}}^{t_{\mathrm{b}}} \Big[a(t)\dot{x}^2 + b(t)\dot{x}x + c(t)x^2 + d(t)\dot{x} + e(t)x + f(t)\Big]\mathrm{d}t. \tag{2.15}$$

We can represent $x(t)$ in terms of $x_{\mathrm{cl}}(t)$ and a new variable y :

$$x(t) = x_{\mathrm{cl}}(t) + y(t),$$

where

$x_{\mathrm{cl}}(t)$ is the classical path,

$y(t)$ is the deviation from the classical path with the conditions that $y(t)$ vanishes at the two end points, $y(t_a) = y(t_b) = 0$.

That is to say, instead of defining a point on the path by its distance $x(t)$ from an arbitrary coordinate axis, we measure the deviation $y(t)$ from the classical path as shown in Fig. 2.4

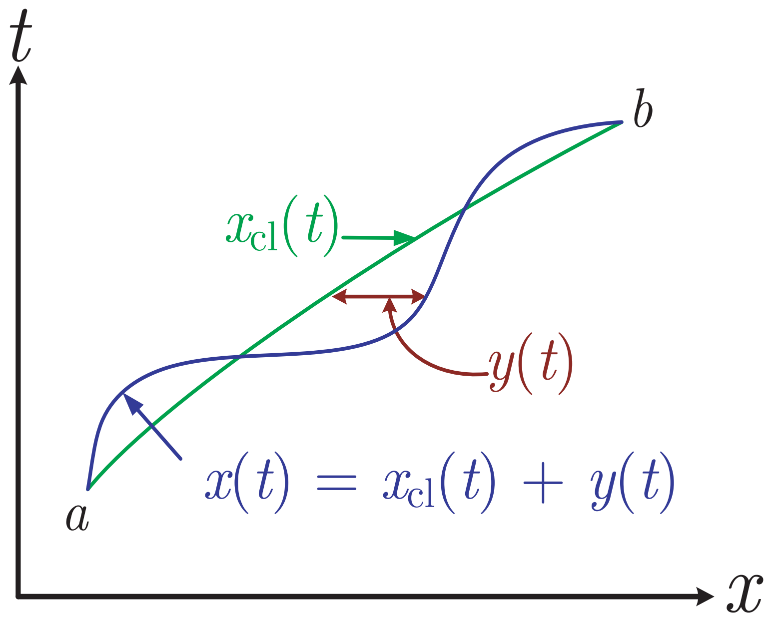

Figure 2.4: The classical paths $x_{\text{cl}}(t)$ and some possible alternative path $x(t) = x_{\text{cl}}(t) + y(t)$.

Substitute $x(t) = x_{\text{cl}}(t) + y(t)$ into equation (2.15)

$$\begin{aligned}
S[x(t)] &= \int_{t_a}^{t_b} \Big[a(t)(\dot{x}_{\text{cl}} + \dot{y})^2 + b(t)(\dot{x}_{\text{cl}} + \dot{y})(x_{\text{cl}} + y) + c(t)(x_{\text{cl}} + y)^2 \\
&\qquad + d(t)(\dot{x}_{\text{cl}} + \dot{y}) + e(t)(x_{\text{cl}} + y) + f(t)\Big]\text{d}t \\
&= \int_{t_a}^{t_b} \Big[a(t)(\dot{x}_{\text{cl}}^2 + 2\dot{x}_{\text{cl}}\dot{y} + \dot{y}^2) + b(t)(\dot{x}_{\text{cl}}x_{\text{cl}} + x_{\text{cl}}\dot{y} + \dot{x}_{\text{cl}}y + \dot{y}y) \\
&\qquad + c(t)(x_{\text{cl}}^2 + 2x_{\text{cl}}y + y^2) + d(t)(\dot{x}_{\text{cl}} + \dot{y}) + e(t)(x_{\text{cl}} + y) + f(t)\Big]\text{d}t \\
&= \int_{t_a}^{t_b} \Big[a(t)\dot{x}_{\text{cl}}^2 + b(t)\dot{x}_{\text{cl}}x_{\text{cl}} + c(t)x_{\text{cl}}^2 + d(t)\dot{x}_{\text{cl}} + e(t)x_{\text{cl}} + f(t)\Big]\text{d}t \\
&\quad + \int_{t_a}^{t_b} \Big[a(t)(2\dot{x}_{\text{cl}}\dot{y} + \dot{y}^2) + b(t)(\dot{x}_{\text{cl}}y + x_{\text{cl}}\dot{y} + \dot{y}y) \\
&\qquad + c(t)(2x_{\text{cl}}y + y^2) + d(t)\dot{y} + e(t)y\Big]\text{d}t \\
S[x(t)] &= S_{\text{cl}} + \int_{t_a}^{t_b} \Big[a(t)(2\dot{x}_{\text{cl}}\dot{y} + \dot{y}^2) + b(t)(\dot{x}_{\text{cl}}y + x_{\text{cl}}\dot{y} + \dot{y}y) \\
&\qquad + c(t)(2x_{\text{cl}}y + y^2) + d(t)\dot{y} + e(t)y\Big]\text{d}t, \qquad (2.16)
\end{aligned}$$

where

$$S_{\rm cl} = \int_{t_{\rm a}}^{t_{\rm b}} \Big[a(t)\dot{x}_{\rm cl}^2 + b(t)\dot{x}_{\rm cl}x_{\rm cl} + c(t)x_{\rm cl}^2 + d(t)\dot{x}_{\rm cl} + e(t)x_{\rm cl} + f(t)\Big]{\rm d}t.$$

If all the term which contain $y(t)$ as a linear factor are collected, the resulting integral vanishes. This could be proved by actually carrying out the integration (involving some integration by parts):

$$\int_{t_{\rm a}}^{t_{\rm b}} a(t)\dot{x}_{\rm cl}\dot{y}{\rm d}t = -\Big[\int_{t_{\rm a}}^{t_{\rm b}} a(t)\ddot{x}_{\rm cl}y{\rm d}t + \int_{t_{\rm a}}^{t_{\rm b}} \dot{a}(t)\dot{x}_{\rm cl}y{\rm d}t\Big] \tag{2.17}$$

$$\int_{t_{\rm a}}^{t_{\rm b}} b(t)x_{\rm cl}\dot{y}{\rm d}t = -\Big[\int_{t_{\rm a}}^{t_{\rm b}} b(t)\dot{x}_{\rm cl}y{\rm d}t + \int_{t_{\rm a}}^{t_{\rm b}} \dot{b}(t)x_{\rm cl}y{\rm d}t\Big] \tag{2.18}$$

$$\int_{t_{\rm a}}^{t_{\rm b}} d(t)\dot{y}{\rm d}t \qquad = -\int_{t_{\rm a}}^{t_{\rm b}} y\dot{d}(t){\rm d}t. \tag{2.19}$$

Substitute equations (2.17), (2.18) and (2.19) into (2.16), we obtain

$$\begin{aligned}
S[x(t)] &= S_{\rm cl} + \int_{t_{\rm a}}^{t_{\rm b}} \Big[-\Big(2a(t)\ddot{x}_{\rm cl}y + 2\dot{a}(t)\dot{x}_{\rm cl}y\Big) + a(t)\dot{y}^2 + b(t)\dot{x}_{\rm cl}y \\
&\quad -\Big(b(t)\dot{x}_{\rm cl}y + \dot{b}(t)x_{\rm cl}y\Big) + b(t)\dot{y}y + 2c(t)x_{\rm cl}y + c(t)y^2 - \dot{d}(t)y + e(t)y\Big]{\rm d}t \\
S[x(t)] &= S_{\rm cl} + \int_{t_{\rm a}}^{t_{\rm b}} \Big[-y\Big(2a(t)\ddot{x}_{\rm cl} + 2\dot{a}(t)\dot{x}_{\rm cl} + \dot{b}(t)x_{\rm cl} - 2c(t)x_{\rm cl} + \dot{d}(t) - e(t)\Big) \\
&\quad +a(t)\dot{y}^2 + b(t)\dot{y}y + c(t)y^2\Big]{\rm d}t.
\end{aligned} \tag{2.20}$$

From the classical Lagrangian equation of motion

$$\frac{\rm d}{{\rm d}t}\Big(\frac{\partial L}{\partial \dot{x}_{\rm cl}}\Big) - \frac{\partial L}{\partial x_{\rm cl}} = 0,$$

$$\begin{aligned}
\frac{\rm d}{{\rm d}t}\Big(2a(t)\dot{x}_{\rm cl} + b(t)x_{\rm cl} + d(t)\Big) - \Big(b(t)\dot{x}_{\rm cl} + 2c(t)x_{\rm cl} + e(t)\Big) &= 0 \\
\Big(2a(t)\ddot{x}_{\rm cl} + 2\dot{a}(t)\dot{x}_{\rm cl} + b(t)\dot{x}_{\rm cl} + \dot{b}(t)x_{\rm cl} + \dot{d}(t)\Big) - \Big(b(t)\dot{x}_{\rm cl} + 2c(t)x_{\rm cl} + e(t)\Big) &= 0 \\
2a(t)\ddot{x}_{\rm cl} + 2\dot{a}(t)\dot{x}_{\rm cl} + \dot{b}(t)x_{\rm cl} - 2c(t)x_{\rm cl} + \dot{d}(t) - e(t) &= 0.
\end{aligned} \tag{2.21}$$

Then using equation (2.20) and the result of equation (2.21), the resulting action is

$$S[x(t)] = S_{\rm cl} + \int_{t_{\rm a}}^{t_{\rm b}} \Big(a(t)\dot{y}^2 + b(t)\dot{y}y + c(t)y^2\Big){\rm d}t. \tag{2.22}$$

Finally substitute equation (2.22) into the equation

$$K(b,a) = \int_{\mathrm{a}}^{\mathrm{b}} \exp\Big\{\frac{i}{\hbar} S[x(t)]\Big\} \mathrm{D}x(t).$$

Since the classical path is completely fixed, any variation in the alternative path $x(t)$ is equivalent to the associated variation in the difference $y(t)$. Thus in a path integral, the path differential $\mathrm{D}x(t)$ can be replaced by $\mathrm{D}y(t)$. Furthermore, the new path variable $y(t)$ is restricted to take the value 0 at both end-points. This substitution leads to a path integral independent of end-point positions, so the propagator in this case can be written as

$$\begin{aligned} K(b,a) &= \int_0^0 \exp\Big\{\frac{i}{\hbar}\Big[S_{\mathrm{cl}} + \int_{t_{\mathrm{a}}}^{t_{\mathrm{b}}} \Big(a(t)\dot{y}^2 + b(t)\dot{y}y + c(t)y^2\Big)\mathrm{d}t\Big]\Big\}\mathrm{D}y(t) \\ &= \exp\Big(\frac{i}{\hbar}S_{\mathrm{cl}}\Big) \int_0^0 \exp\Big[\frac{i}{\hbar}\int_{t_{\mathrm{a}}}^{t_{\mathrm{b}}} \Big(a(t)\dot{y}^2 + b(t)\dot{y}y + c(t)y^2\Big)\mathrm{d}t\Big]\mathrm{D}y(t). \end{aligned}$$

Therefore

$$K(b,a) = F(t_{\mathrm{b}}, t_{\mathrm{a}}) \exp\Big(\frac{i}{\hbar}S_{\mathrm{cl}}\Big), \tag{2.23}$$

where

$$F(t_{\mathrm{b}}, t_{\mathrm{a}}) = \int_0^0 \exp\Big[\frac{i}{\hbar}\int_{t_a}^{t_{\mathrm{b}}} \Big(a(t)\dot{y}^2 + b(t)\dot{y}y + c(t)y^2\Big)\mathrm{d}t\Big]\mathrm{D}y(t),$$

and $F(t_{\mathrm{b}}, t_{\mathrm{a}})$ is so called *prefactor*.

2.4 The free-particle propagator

Consider a free particle which has Lagrangian

$$L = \frac{1}{2}m\dot{x}^2. \tag{2.24}$$

The equation of motion can be found from Lagrange's equation as follows

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{x}}\Big) - \frac{\partial L}{\partial x} &= 0 \\ m\ddot{x} &= 0 \\ \ddot{x} &= 0. \end{aligned} \tag{2.25}$$

From equation (2.25), the solution is

$$x_{\mathrm{cl}} = Ct + C', \tag{2.26}$$

where C and C' are constants.
From Fig. (2.1) and the boundary conditions

$$\begin{aligned} x_{\rm cl}(t_a) &= x_a, \\ x_{\rm cl}(t_b) &= x_b. \end{aligned}$$

We obtain

$$x_a = Ct_a + C', \tag{2.27}$$

$$x_b = Ct_b + C'. \tag{2.28}$$

We can solve for C and C' easily, it yields

$$\begin{aligned} C &= \frac{x_b - x_a}{t_b - t_a}. \\ C' &= \frac{x_a t_b - x_b t_a}{t_b - t_a}, \end{aligned}$$

substitute C and C' into equation (2.26), we get

$$x_{\rm cl} = \left(\frac{x_b - x_a}{t_b - t_a}\right)t + \left(\frac{x_a t_b - x_b t_a}{t_b - t_a}\right), \tag{2.29}$$

and

$$\dot{x}_{\rm cl} = \left(\frac{x_b - x_a}{t_b - t_a}\right).$$

Therefore the action of the free particle can be evaluated as follows

$$S_{\rm cl} = \int_{t_a}^{t_b} \frac{m}{2}\dot{x}_{\rm cl}^2 {\rm d}t.$$

Integrating by part, we get

$$S_{\rm cl} = \frac{m}{2}\Big(x_{\rm cl}(t_b)\dot{x}_{\rm cl}(t_b) - x_{\rm cl}(t_a)\dot{x}_{\rm cl}(t_a)\Big) \tag{2.30}$$

inserting $x_{\rm cl}, \dot{x}_{\rm cl}$ into equation (2.30) therefore

$$S_{\rm cl} = \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_a}. \tag{2.31}$$

From the relation

$$K(x_b, t_b; x_a, t_a) = \int_{-\infty}^{\infty} K(x_b, t_b; x_{\rm c}, t_{\rm c})K(x_{\rm c}, t_{\rm c}; x_a, t_a){\rm d}x_{\rm c},$$

substituting the free particle propagator into the equation above, it becomes

$$F(t_b,t_a)\exp\left\{\frac{mi(x_b-x_a)^2}{2\hbar(t_b-t_a)}\right\}=F(t_b,t_c)F(t_c,t_a)\int_{-\infty}^{\infty}\exp\left\{\left[\frac{mi}{2\hbar}\frac{(x_b-x_c)^2}{(t_b-t_c)}+\frac{(x_c-x_a)^2}{(t_c-t_a)}\right]\right\}\mathrm{d}x_c.$$

We are interested in only $F(t_b,t_a)$, therefore we set $x_a=0,\ x_b=0$ for convenience then

$$\begin{aligned}F(t_b,t_a) &= F(t_b,t_\mathrm{c})F(t_\mathrm{c},t_a)\int_{-\infty}^{\infty}\exp\left\{\frac{mi}{2\hbar}\left[\frac{x_\mathrm{c}^2}{(t_\mathrm{b}-t_\mathrm{c})}+\frac{x_\mathrm{c}^2}{(t_\mathrm{c}-t_\mathrm{a})}\right]\right\}\mathrm{d}x_\mathrm{c}\\ F(t_b,t_a) &= F(t_b,t_\mathrm{c})F(t_\mathrm{c},t_a)\int_{-\infty}^{\infty}\exp\left\{\frac{mi}{2\hbar}\left[\left(\frac{t_\mathrm{b}-t_\mathrm{a}}{(t_\mathrm{b}-t_\mathrm{c})(t_\mathrm{c}-t_\mathrm{a})}\right)x_\mathrm{c}^2\right]\right\}\mathrm{d}x_\mathrm{c}.\end{aligned} \tag{2.32}$$

By performing Gaussian integral, it gives

$$\begin{aligned}F(t_b,t_a) &= F(t_b,t_\mathrm{c})F(t_\mathrm{c},t_a)\left(\frac{2\pi i\hbar}{m}\frac{(t_b-t_\mathrm{c})(t_\mathrm{c}-t_a)}{(t_b-t_a)}\right)^{1/2}\\ \frac{F(t_b,t_a)}{F(t_b,t_\mathrm{c})F(t_\mathrm{c},t_a)} &= \sqrt{\frac{2\pi i\hbar}{m}\frac{(t_b-t_\mathrm{c})(t_\mathrm{c}-t_a)}{(t_b-t_a)}}\\ \frac{F(t_b,t_a)}{F(t_b,t_\mathrm{c})F(t_\mathrm{c},t_a)} &= \sqrt{\frac{\frac{2\pi i\hbar}{m}(t_b-t_\mathrm{c})\frac{2\pi i\hbar}{m}(t_\mathrm{c}-t_a)}{\frac{2\pi i\hbar}{m}(t_b-t_a)}}.\end{aligned}$$

By comparing the prefactor term by term, we conclude that

$$F(t_b,t_a)=\sqrt{\frac{m}{2\pi i\hbar(t_b-t_a)}}.$$

Therefore, the free particle propagator is

$$K(x_b,t_b;x_a,t_a)=\sqrt{\frac{m}{2\pi i\hbar(t_b-t_a)}}\exp\left\{\frac{mi}{2\hbar}\frac{(x_\mathrm{b}-x_\mathrm{a})^2}{(t_\mathrm{b}-t_\mathrm{a})}\right\}. \tag{2.33}$$

Chapter 3

Green's functions

In general we can solve the inhomogeneous differential equation in many ways. For instance, the method of undetermined coefficients and variation of parameters. In this chapter, we turn our attention to another method so called *Green's function.* It can be useful in both classical physics and quantum mechanics. Suppose that we have the equation

$$\hat{L}y \quad = \quad f, \tag{3.1}$$

where

$\hat{L}$ is some linear ordinary differential operator,

f is a given function of x. Throwing aside questions of rigor, suppose now that $\hat{L}$ possesses a complete orthonomal set of eigenfunctions $\phi_n(x)$ so that

$$\hat{L}\phi_n(x) \quad = \quad \lambda_n\phi_n(x), \tag{3.2}$$

for any function $g(x)$, it can be expanded in term of $\phi_n(x)$ as follows

$$\begin{aligned} g(x) &= c_1\phi_1(x) + c_2\phi_2(x) + ... + c_n\phi_n(x) + ... \\ g(x) &= \sum_{n=1}^{\infty} c_n\phi_n(x), \end{aligned} \tag{3.3}$$

where c_n is an expansion coefficient.

We multiply equation (3.3) by $\phi^*_m(x)$ and integrate with respect to x from $-\infty$ to ∞, so that

$$\begin{aligned}\int_{-\infty}^{\infty} \phi_m^*(x) g(x) \mathrm{d}x &= \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} c_n \phi_m^*(x) \phi_n(x) \mathrm{d}x \\ &= \sum_{n=1}^{\infty} c_n \int_{-\infty}^{\infty} \phi_m^*(x) \phi_n(x) \mathrm{d}x \\ &= c_1 \int_{-\infty}^{\infty} \phi_m^*(x) \phi_1(x) \mathrm{d}x + c_2 \int_{-\infty}^{\infty} \phi_m^*(x) \phi_2(x) \mathrm{d}x + ... \\ &\quad + c_m \int_{-\infty}^{\infty} \phi_m^*(x) \phi_m(x) \mathrm{d}x + ... + c_n \int_{-\infty}^{\infty} \phi_m^*(x) \phi_n(x) \mathrm{d}x + ...\end{aligned}$$

From $\int_{-\infty}^{\infty} \phi_m^*(x)\phi_n(x)\mathrm{d}x \;=\; \delta_{mn}$, we conclude that

$$\int_{-\infty}^{\infty} \phi_m^*(x) g(x) \mathrm{d}x = c_m. \tag{3.4}$$

Under circumstance of equation (3.2), we write

$$y(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x), \tag{3.5}$$

$$f(x) = \sum_{n=1}^{\infty} \beta_n \phi_n(x). \tag{3.6}$$

Substitute equations (3.5) and (3.6) into equation (3.1), we obtain

$$\begin{aligned}\hat{L}\left[\sum_{n=1}^{\infty} \alpha_n \phi_n(x)\right] &= \sum_{n=1}^{\infty} \beta_n \phi_n(x) \\ \sum_{n=1}^{\infty} \alpha_n \hat{L} \phi_n(x) &= \sum_{n=1}^{\infty} \beta_n \phi_n(x) \\ \sum_{n=1}^{\infty} \alpha_n \lambda_n \phi_n(x) &= \sum_{n=1}^{\infty} \beta_n \phi_n(x) \\ \sum_{n=1}^{\infty} \Big(\alpha_n \lambda_n - \beta_n\Big) \phi_n(x) &= 0\end{aligned}$$

$\phi_n(x) \neq 0,$ so that

$$\alpha_n \lambda_n - \beta_n = 0.$$

Thus

$$\alpha_n = \frac{\beta_n}{\lambda_n}. \tag{3.7}$$

If we now substitute equation (3.7) into equation (3.5), we obtain

$$y(x) = \sum_{n=1}^{\infty} \frac{\beta_n}{\lambda_n} \phi_n(x). \tag{3.8}$$

We may rewrite equation (3.8) as

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\int_{-\infty}^{\infty} \phi_n^*(x') f(x') \mathrm{d}x' \right) \phi_n(x) \\ y(x) &= \int_{-\infty}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n} \phi_n(x) \phi_n^*(x') \right) f(x') \mathrm{d}x'. \end{aligned}$$

So

$$y(x) = \int_{-\infty}^{\infty} G(x, x') f(x') \mathrm{d}x', \tag{3.9}$$

where

$$G(x, x') = \sum_{n=1}^{\infty} \frac{\phi_n(x) \phi_n^*(x')}{\lambda_n}. \tag{3.10}$$

This the *Green's function*.

Thus we obtain a representation of the Green's function in term of an infinite eigenfunction expansion. Now we want to rewrite it in closed form.
An interesting way of viewing the Green's function is to operate on $G(x, x')$ with $\hat{L}$:

$$\hat{L}\, G(x, x') = \hat{L} \sum_{n=1}^{\infty} \frac{\phi_n(x) \phi_n^*(x')}{\lambda_n}.$$

We use $\hat{L}\phi_n(x) = \lambda_n \phi_n(x)$, then

$$\begin{aligned} \hat{L} G(x, x') &= \sum_{n=1}^{\infty} \hat{L} \frac{\phi_n(x) \phi_n^*(x')}{\lambda_n} \\ &= \sum_{n=1}^{\infty} \phi_n(x) \phi_n^*(x') \quad \equiv \quad I(x, x'). \end{aligned} \tag{3.11}$$

For any function $f(x)$;

$$\begin{aligned} \int_{-\infty}^{\infty} I(x, x') f(x') \mathrm{d}x' &= \sum_{n=1}^{\infty} \phi_n(x) \int_{-\infty}^{\infty} \phi_n^*(x') f(x') \mathrm{d}x'. \\ &= \sum_{n=1}^{\infty} \beta_n \phi_n(x), \end{aligned} \tag{3.12}$$

this equation may be written as

$$\int_{-\infty}^{\infty} I(x,x')f(x')\mathrm{d}x' = f(x). \tag{3.13}$$

The function $I(x,x')$ with this property is just the Dirac δ - function, $\delta(x-x')$. Thus the Green's function can be obtained from the equation

$$\hat{L}G(x,x') = \delta(x-x'). \tag{3.14}$$

Let us consider the most general second - order linear differential operator

$$\hat{L} \equiv f_0(t)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + f_1(t)\frac{\mathrm{d}}{\mathrm{d}t} + f_2(t), \tag{3.15}$$

where f_0, f_1, and f_2 are functions of t.
We want to find $x(t)$ according to

$$\hat{L}x(t) = F(t).$$

Let $x_1(t)$ and $x_2(t)$ be two linearly independent solutions to

$$\hat{L}x(t) = 0.$$

We wish to solve

$$\begin{aligned} \hat{L}G(t,t') &= f_0(t)\frac{\mathrm{d}^2G(t,t')}{\mathrm{d}t^2} + f_1(t)\frac{\mathrm{d}G(t,t')}{\mathrm{d}t} + f_2(t)G(t,t') \\ &= \delta(t-t'). \end{aligned} \tag{3.16}$$

For $t > t'$, we write

$$G(t,t') = a_1x_1(t) + a_2x_2(t), \tag{3.17}$$

and for $t < t'$,

$$G(t,t') = b_1x_1(t) + b_2x_2(t). \tag{3.18}$$

Now let us try to determine a_1, a_2, b_1, and b_2. At $t = t'$, $G(t,t')$ must be continuous; if it were not, $\mathrm{d}G/\mathrm{d}t$ would contain a δ - function and therefore $\mathrm{d}^2G/\mathrm{d}t^2$ would contain the derivative of a δ - function. However on the right - hand side of equation (3.16) there is just a δ - function, so we conclude that $G(t,t')$ is continuous at $t = t'$,

$\mathrm{d}G/\mathrm{d}t$ will not be continuous at $t = t'$, and from equation (3.16) we can determine the value of

$$\left[\frac{\mathrm{d}G(t,t')}{\mathrm{d}t}\right]_{t=t'+\epsilon} - \left[\frac{\mathrm{d}G(t,t')}{\mathrm{d}t}\right]_{t=t'-\epsilon}$$

by integrating both sides of equation (3.16) from $t = t' - \epsilon$ to $t = t' + \epsilon$.
On the right - hand side, the δ - function yields 1, so

$$\int_{t'-\epsilon}^{t'+\epsilon} f_0(t)\frac{\mathrm{d}^2G(t,t')}{\mathrm{d}t^2}\mathrm{d}t + \int_{t'-\epsilon}^{t'+\epsilon} f_1(t)\frac{\mathrm{d}G(t,t')}{\mathrm{d}t}\mathrm{d}t + \int_{t'-\epsilon}^{t'+\epsilon} f_2(t)G(t,t')\mathrm{d}t = 1.$$

Let us assume that $f_0, f_1,$ and f_2 are continuous throughout the region of definition of the Green's function. Since ϵ is very small, these functions vary negligibly in the region of integration. We can replace them with their values at $t = t'$:

$$f_0(t')\int_{t'-\epsilon}^{t'+\epsilon} \frac{\mathrm{d}^2G(t,t')}{\mathrm{d}t^2}\mathrm{d}t + f_1(t')\int_{t'-\epsilon}^{t'+\epsilon} \frac{\mathrm{d}G(t,t')}{\mathrm{d}t}\mathrm{d}t + f_2(t')\int_{t'-\epsilon}^{t'+\epsilon} G(t,t')\mathrm{d}t = 1.$$

Since G is continuous at $t = t'$ and the domain of integration can be made arbitrarily small, the last term on the left-hand side of this equation vanishes. Performing the remaining two integrals, we get

$$f_0(t')\left\{\left[\frac{\mathrm{d}G(t,t')}{\mathrm{d}t}\right]_{t=t'+\epsilon} - \left[\frac{\mathrm{d}G(t,t')}{\mathrm{d}t}\right]_{t=t'-\epsilon}\right\} + f_1(t')\Big\{G(t'+\epsilon,t') - G(t'-\epsilon,t')\Big\} = 1.$$

Since $G(t,t')$ is continuous at $t = t'$, the second term on the left - hand side vanishes. We are left with

$$\left[\frac{\mathrm{d}G(t,t')}{\mathrm{d}t}\right]_{t=t'+\epsilon} - \left[\frac{\mathrm{d}G(t,t')}{\mathrm{d}t}\right]_{t=t'-\epsilon} = \frac{1}{f_0(t')}. \tag{3.19}$$

Using this relation and the fact that $G(t,t')$ is continuous at $t = t'$, we can obtain useful information about the coefficients $a_1, a_2, b_1,$ and b_2 in equations (3.17) and (3.18). From continuity at $t = t'$, we get

$$a_1x_1(t') + a_2x_2(t') = b_1x_1(t') + b_2x_2(t'),$$

while from equation (3.19), we obtain

$$a_1\dot{x}_1(t') + a_2\dot{x}_2(t') - [b_1\dot{x}_1(t') + b_2\dot{x}_2(t')] = \frac{1}{f_0(t')}.$$

Combining terms, we find the following set of equations for the differences $(a_1 - b_1)$ and $(a_2 - b_2)$:

$$\begin{aligned}(a_1 - b_1)x_1(t') + (a_2 - b_2)x_2(t') &= 0, \\ (a_1 - b_1)\dot{x}_1(t') + (a_2 - b_2)\dot{x}_2(t') &= \frac{1}{f_0(t')}.\end{aligned}$$

The solutions are

$$a_1 - b_1 = -\frac{x_2(t')}{f_0(t')W(t')} \quad , \quad a_2 - b_2 = \frac{x_1(t')}{f_0(t')W(t')},$$

$W(t')$ is the *Wronskian* of x_1 and x_2, defined by

$$\begin{aligned}W(t') &= x_1(t')\dot{x}_2(t') - \dot{x}_1(t')x_2(t') \\ &= \begin{vmatrix} x_1(t') & x_2(t') \\ \dot{x}_1(t') & \dot{x}_2(t') \end{vmatrix} .\end{aligned}$$

In this determinant form, the Wronskian generalizes differential equations of any order. Putting the above result into equations (3.17) and (3.18), we see that $G(t,t')$ may be written as

$$G(t,t') = b_1x_1(t) + b_2x_2(t) - \left(\frac{x_1(t)x_2(t') - x_2(t)x_1(t')}{f_0(t')W(t')}\right), \qquad t > t' \quad (3.20)$$

$$G(t,t') = b_1x_1(t) + b_2x_2(t), \quad t < t' \quad (3.21)$$

The two remaining constants, b_1 and b_2, can now be chosen to satisfy the appropriate boundary conditions. The form which the Green's function finally takes is strongly dependent on the type of boundary conditions involved. If the boundary conditions are of the single point variety typical of classical mechanics, then we shall require that $b_1 = 0 = b_2$ in equations (3.20) and (3.21). If we do this, the Green's function is given by

$$G(t,t') = -\frac{x_1(t)x_2(t') - x_2(t)x_1(t')}{f_0(t')W(t')} \quad , \quad (3.22)$$

and the solution in term of the Green's function is

$$x(t) = Ax_1(t) + Bx_2(t) + \int_{t_0}^{t} G(t,t')F(t')\mathrm{d}t'. \quad (3.23)$$

The term involving the integral vanishes at $t = t_0$ and so does its derivative with respect to t. Thus, when we apply the one - point boundary conditions, $x(t_0) = x_0$,

$\dot{x}(t_0) = \dot{x}_0$, the constants A and B in equation (3.23) are determined just as though the integral term were not present.
However if the boundary conditions are of the two - point type, say

$$x(t_0) = x_0 \ \ , \ \ x(t_1) = x_1,$$

then we write

$$x(t) = Ax_1(t) + Bx_2(t) + \int_{t_0}^{t_1} G(t,t')F(t')\mathrm{d}t'. \tag{3.24}$$

Then it would seem reasonable to choose b_1 and b_2 in equations (3.20) and (3.21) so that $G(t_0, t') = 0 = G(t_1, t')$ and therefore A and B of equation (3.24) are determined by the boundary condition, just as if the integral term does not present.

Chapter 4

The forced harmonic oscillator propagator

4.1 The classical action of the forced harmonic oscillator

In this section we go back to study the single harmonic oscillator, but coupled linearly to some external potential or disturbance. The Lagrangian for such a system is given by

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 + f(t)x, \tag{4.1}$$

where $f(t)$ is the external force exerting on the particle of mass m.
The equation of motion of this particle can be found from Lagrange's equation, we get

$$\ddot{x} + \omega^2 x = \frac{f(t)}{m}. \tag{4.2}$$

The general solution is $x = x_{\mathrm{c}} + x_{\mathrm{p}}$, where x_{c} is the solution of the homogeneous equation

$$\ddot{x} + \omega^2 x = 0. \tag{4.3}$$

The solution is of the form

$$x_{\mathrm{c}} = A\cos\omega t + B\sin\omega t, \tag{4.4}$$

where A and B are arbitrary constants.
The other solution, x_{p}, is solved from the inhomogeneous equation

$$\ddot{x} + \omega^2 x = \frac{f(t)}{m}.$$

By Green's function method, x_{p} can be written as

$$x_{\mathrm{p}}(t) = \int_{t_1}^{t_2} G(t,t')\frac{f(t')}{m}\mathrm{d}t', \tag{4.5}$$

where $G(t,t')$ is the Green's function.

From

$$G(t,t') = b_1 x_1(t) + b_2 x_2(t) - \left(\frac{x_1(t)x_2(t') - x_2(t)x_1(t')}{f_0(t')W(t')}\right),\ t > t' \tag{4.6}$$

$$G(t,t') = b_1 x_1(t) + b_2 x_2(t),\ t < t', \tag{4.7}$$

with the boundary conditions

$$G(t_a,t') = 0 = G(t_b,t'), \tag{4.8}$$

and where x_1 and x_2 satisfy the homogeneous equation mentioned in equation (4.4) and $W(t')$ is the Wronskian defined by

$$\begin{aligned} W(t') &= \begin{vmatrix} x_1(t') & x_2(t') \\ \dot{x}_1(t') & \dot{x}_2(t') \end{vmatrix} \\ &= \begin{vmatrix} \cos\omega t' & \sin\omega t' \\ -\omega\sin\omega t' & \omega\cos\omega t' \end{vmatrix} \\ W(t') &= \omega, \end{aligned}$$

then

$$\begin{aligned} G(t,t') &= b_1\cos\omega t + b_2\sin\omega t - \left[\frac{(\cos\omega t)(\sin\omega t') - (\sin\omega t)(\cos\omega t')}{\omega}\right], \qquad t > t' \\ G(t,t') &= b_1\cos\omega t + b_2\sin\omega t, \qquad t < t'. \end{aligned}$$

We obtain

$$G(t,t') = b_1\cos\omega t + b_2\sin\omega t + \frac{1}{\omega}\sin\omega(t-t'),\ t > t' \tag{4.9}$$

$$G(t,t') = b_1\cos\omega t + b_2\sin\omega t,\ t < t'. \tag{4.10}$$

b_1 and b_2 can be found from the boundary condition (4.8).
At $t = t_a$;

$$G(t_a, t') = b_1 \cos \omega t_a + b_2 \sin \omega t_a = 0 \tag{4.11}$$

$$b_1 = -b_2 \frac{\sin \omega t_a}{\cos \omega t_a} \tag{4.12}$$

At $t = t_b$;

$$G(t_b, t') = b_1 \cos \omega t_b + b_2 \sin \omega t_b + \frac{1}{\omega} \sin \omega (t_b - t') = 0. \tag{4.13}$$

Substituting equation (4.12) into equation (4.13), the result is

$$b_2 = -\frac{1}{\omega} \frac{\sin \omega (t_b - t') \cos \omega t_a}{\sin \omega (t_b - t_a)}.$$

Substituting b_2 back into equation (4.12), we get

$$b_1 = \frac{1}{\omega} \frac{\sin \omega t_a \sin \omega (t_b - t')}{\sin \omega (t_b - t_a)}.$$

When b_1 and b_2 are put in equations (4.9) and (4.10), the final forms of the Green's functions are

$$G(t, t') = \frac{1}{\omega} \frac{\sin \omega (t_b - t')}{\sin \omega (t_b - t_a)} \sin \omega (t_a - t) + \frac{1}{\omega} \sin \omega (t - t'), \; t > t' \tag{4.14}$$

$$G(t, t') = \frac{1}{\omega} \frac{\sin \omega (t_b - t')}{\sin \omega (t_b - t_a)} \sin \omega (t_a - t), \; t < t'. \tag{4.15}$$

From equations (4.4) and (4.5)

$$x_{\rm cl}(t) = \mathrm{A} x_1(t) + \mathrm{B} x_2(t) + \int_{t_a}^{t_b} G(t, t') F(t') \mathrm{d}t'$$

$$x_{\rm cl}(t) = \mathrm{A} \cos \omega t + \mathrm{B} \sin \omega t + \int_{t_a}^{t} G(t, t') F(t') \mathrm{d}t' + \int_{t}^{t_b} G(t, t') F(t') \mathrm{d}t'. \tag{4.16}$$

Consider $\int_{t_a}^{t} G(t, t') F(t') \mathrm{d}t'$, it's obvious that we have to choose $G(t, t')$ at $t > t'$

$$\int_{t_a}^{t} G(t, t') F(t') \mathrm{d}t' = \int_{t_a}^{t} \left[\frac{1}{\omega} \frac{\sin \omega (t_b - t')}{\sin \omega (t_b - t_a)} \sin \omega (t_a - t) + \frac{1}{\omega} \sin \omega (t - t') \right] \frac{f(t')}{m} \mathrm{d}t'.$$

$$\int_{t_a}^{t} G(t, t') F(t') \mathrm{d}t' = \frac{1}{m\omega} \frac{\sin \omega (t_a - t)}{\sin \omega (t_b - t_a)} \int_{t_a}^{t} \sin \omega (t_b - t') f(t') \mathrm{d}t' + \frac{1}{m\omega} \int_{t_a}^{t} \sin \omega (t - t') f(t') \mathrm{d}t',$$

and the suitable Green's function for $\int_{t}^{t_b} G(t, t') F(t') \mathrm{d}t'$ is $G(t, t')$ at $t < t'$, so

$$\int_{t}^{t_b} G(t, t') F(t') \mathrm{d}t' = \frac{1}{m\omega} \frac{\sin \omega (t_a - t)}{\sin \omega (t_b - t_a)} \int_{t}^{t_b} \sin \omega (t_b - t') f(t') \mathrm{d}t'.$$

Finally we get

$$\begin{aligned} x_{\rm cl}(t) &= A\cos\omega t + B\sin\omega t + \frac{1}{m\omega}\frac{\sin\omega(t_a-t)}{\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t_b-t')f(t')\mathrm{d}t' \\ &\quad +\frac{1}{m\omega}\int_{t_a}^{t}\sin\omega(t-t')f(t')\mathrm{d}t'. \end{aligned} \tag{4.17}$$

We can find A, B from the two boundary conditions

$$x_{\rm cl}(t_a) = x_a \quad , \quad x_{\rm cl}(t_b) = x_b.$$

From equation (4.17), we obtain

$$x_a = A\cos\omega t_a + B\sin\omega t_a \tag{4.18}$$

$$x_b = A\cos\omega t_b + B\sin\omega t_b. \tag{4.19}$$

From Kramer's rule, we can write

$$A = \frac{\begin{vmatrix} x_{\rm a} & \sin\omega t_{\rm a} \\ x_{\rm b} & \sin\omega t_{\rm b} \end{vmatrix}}{\begin{vmatrix} \cos\omega t_{\rm a} & \sin\omega t_{\rm a} \\ \cos\omega t_{\rm b} & \sin\omega t_{\rm b} \end{vmatrix}} = \frac{x_a\sin\omega t_b - x_b\sin\omega t_a}{\sin\omega(t_b-t_a)} \tag{4.20}$$

$$B = \frac{\begin{vmatrix} \cos\omega t_{\rm a} & x_{\rm a} \\ \cos\omega t_{\rm b} & x_{\rm b} \end{vmatrix}}{\begin{vmatrix} \cos\omega t_{\rm a} & \sin\omega t_{\rm a} \\ \cos\omega t_{\rm b} & \sin\omega t_{\rm b} \end{vmatrix}} = \frac{x_b\cos\omega t_a - x_a\cos\omega t_b}{\sin\omega(t_b-t_a)}. \tag{4.21}$$

Substituting A and B into equation (4.17), the result is

$$\begin{aligned} x_{\rm cl}(t) &= \left(\frac{x_a\sin\omega t_b - x_b\sin\omega t_a}{\sin\omega(t_b-t_a)}\right)\cos\omega t + \left(\frac{x_b\cos\omega t_a - x_a\cos\omega t_b}{\sin\omega(t_b-t_a)}\right)\sin\omega t \\ &\quad +\frac{1}{m\omega}\frac{\sin\omega(t_a-t)}{\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t_b-t')f(t')\mathrm{d}t' \\ &\quad +\frac{1}{m\omega}\int_{t_a}^{t}\sin\omega(t-t')f(t')\mathrm{d}t' \\ x_{\rm cl}(t) &= \frac{1}{\sin\omega(t_b-t_a)}\Big(x_a\sin\omega(t_b-t) + x_b\sin\omega(t-t_a)\Big) \\ &\quad +\frac{1}{m\omega}\left[\frac{\sin\omega(t_a-t)}{\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t_b-t')f(t')\mathrm{d}t' \right. \\ &\qquad \left. +\int_{t_a}^{t}\sin\omega(t-t')f(t')\mathrm{d}t'\right]. \end{aligned} \tag{4.22}$$

The velocity can be obtained by differentiating $x_{\rm cl}$ with respect to time t

$$\begin{aligned}
\dot{x}_{\rm cl}(t) &= \frac{\omega}{\sin\omega(t_b-t_a)}\Big(x_b\cos\omega(t-t_a)-x_a\cos\omega(t_b-t)\Big)\\
&\quad -\frac{1}{m}\frac{\cos\omega(t_a-t)}{\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t_b-t')f(t'){\rm d}t'\\
&\quad +\frac{1}{m\omega}\frac{\rm d}{{\rm d}t}\bigg(\int_{t_a}^{t}\sin\omega(t-t')f(t'){\rm d}t'\bigg). \qquad (4.23)
\end{aligned}$$

We can differentiate the last term of equation (4.23) by applying *Leibnitz's rule* for differentiating an integral as follows;
If

$$I(\alpha)=\int_{a(\alpha)}^{b(\alpha)}f(x,\alpha){\rm d}x,$$

then

$$\frac{{\rm d}I}{{\rm d}\alpha}=\int_a^b\frac{\partial f}{\partial\alpha}{\rm d}x+f(b,\alpha)\frac{{\rm d}b}{{\rm d}\alpha}-f(a,\alpha)\frac{{\rm d}a}{{\rm d}\alpha}.$$

Let

$$I(t)=\int_{t_a}^{t}\sin\omega(t-t')f(t'){\rm d}t',$$

therefore

$$\frac{{\rm d}I}{{\rm d}t}=\int_{t_a}^{t}\omega\cos\omega(t-t')f(t'){\rm d}t'.$$

Substituting our result into equation (4.23), it gives

$$\begin{aligned}
\dot{x}_{\rm cl}(t) &= \frac{\omega}{\sin\omega(t_b-t_a)}\Big(x_b\cos\omega(t-t_a)-x_a\cos\omega(t_b-t)\Big)\\
&\quad -\frac{1}{m}\frac{\cos\omega(t_a-t)}{\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t_b-t')f(t'){\rm d}t'\\
&\quad +\frac{1}{m}\int_{t_a}^{t}\cos\omega(t-t')f(t'){\rm d}t'. \qquad (4.24)
\end{aligned}$$

Now we are ready to calculate the classical action of the forced harmonic oscillator. We begin with

$$\begin{aligned}
S_{\rm cl} &= \int_{t_a}^{t_b}\bigg(\frac{1}{2}m\dot{x}_{\rm cl}^2-\frac{1}{2}m\omega^2x_{\rm cl}^2+f(t)x_{\rm cl}\bigg){\rm d}t\\
S_{\rm cl} &= \int_{t_a}^{t_b}\frac{1}{2}m\dot{x}_{\rm cl}^2{\rm d}t-\int_{t_a}^{t_b}\frac{1}{2}m\omega^2x_{\rm cl}^2{\rm d}t+\int_{t_a}^{t_b}f(t)x_{\rm cl}{\rm d}t. \qquad (4.25)
\end{aligned}$$

Consider the first term of equation (4.25). By performing integration by part, we get

$$\int_{t_a}^{t_b}\frac{1}{2}m\dot{x}_{\rm cl}^2{\rm d}t=\frac{m}{2}\bigg(x_{\rm cl}\dot{x}_{\rm cl}\Big|_{t_a}^{t_b}-\int_{t_a}^{t_b}x_{\rm cl}\ddot{x}_{\rm cl}{\rm d}t\bigg).$$

Equation (4.25) becomes

$$\begin{aligned} S_{\rm cl} &= \frac{m}{2}x_{\rm cl}\dot{x}_{\rm cl}\Big|_{t_a}^{t_b} - \frac{m}{2}\int_{t_a}^{t_b} x_{\rm cl}\ddot{x}_{\rm cl}{\rm d}t - \frac{m}{2}\int_{t_a}^{t_b}\omega^2 x_{\rm cl}^2{\rm d}t + \int_{t_a}^{t_b} f(t)x_{\rm cl}{\rm d}t \\ &= \frac{m}{2}x_{\rm cl}\dot{x}_{\rm cl}\Big|_{t_a}^{t_b} - \frac{m}{2}\int_{t_a}^{t_b} x_{\rm cl}(\ddot{x}_{\rm cl}+\omega^2 x_{\rm cl}){\rm d}t + \int_{t_a}^{t_b} f(t)x_{\rm cl}{\rm d}t \\ &= \frac{m}{2}x_{\rm cl}\dot{x}_{\rm cl}\Big|_{t_a}^{t_b} - \frac{m}{2}\int_{t_a}^{t_b} x_{\rm cl}\frac{f(t)}{m}{\rm d}t + \int_{t_a}^{t_b} f(t)x_{\rm cl}{\rm d}t. \end{aligned}$$

Therefore

$$S_{\rm cl} = \frac{m}{2}\Big(x_{\rm cl}(t_b)\dot{x}_{\rm cl}(t_b) - x_{\rm cl}(t_a)\dot{x}_{\rm cl}(t_a)\Big) + \frac{1}{2}\int_{t_a}^{t_b} f(t)x_{\rm cl}(t){\rm d}t. \tag{4.26}$$

Replace $x_{\rm cl}, \dot{x}_{xl}$ of equation (4.26) with those of equations (4.22) and (4.24)

$$\begin{aligned} S_{\rm cl} &= \frac{m}{2}\Bigg\{\Bigg[\frac{\omega}{\sin\omega(t_b-t_a)}\Big(x_b^2\cos\omega(t_b-t_a) - x_b x_a\Big) + \frac{x_b}{m}\int_{t_a}^{t_b}\cos\omega(t_b-t')f(t'){\rm d}t' \\ &\quad -\frac{x_b}{m}\frac{\cos\omega(t_a-t_b)}{\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t_b-t')f(t'){\rm d}t'\Bigg] \\ &\quad -\Bigg[\frac{\omega}{\sin\omega(t_b-t_a)}\Big(x_b x_a - x_a^2\cos\omega(t_b-t_a)\Big) \\ &\quad -\frac{x_a}{m\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t_b-t')f(t'){\rm d}t'\Bigg]\Bigg\} \\ &\quad +\Bigg\{\frac{1}{2\sin\omega(t_b-t_a)}\Bigg[x_a\int_{t_a}^{t_b}\sin\omega(t_b-t)f(t){\rm d}t + x_b\int_{t_a}^{t_b}\sin\omega(t-t_a)f(t){\rm d}t\Bigg] \\ &\quad +\frac{1}{2m\omega\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t_b-t')f(t'){\rm d}t'\int_{t_a}^{t_b}\sin\omega(t_a-t)f(t){\rm d}t \\ &\quad +\frac{1}{2m\omega}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t-t')f(t')f(t){\rm d}t'{\rm d}t\Bigg\} \end{aligned}$$

$$
\begin{aligned}
S_{\rm cl} \;=\;& \frac{m\omega}{2\sin\omega(t_b-t_a)}\Big(x_b^2+x_a^2\Big)\Big(\cos\omega(t_b-t_a)\Big)\\
&-\frac{m\omega}{2\sin\omega(t_b-t_a)}\Big(2x_bx_a\Big)\\
&+\frac{x_a}{\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t_b-t)f(t)\mathrm{d}t\\
&+\frac{x_b}{2}\int_{t_a}^{t_b}\cos\omega(t_b-t')f(t')\mathrm{d}t'\\
&-\frac{x_b}{2}\frac{\cos\omega(t_b-t_a)}{\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t_b-t')f(t')\mathrm{d}t'\\
&+\frac{x_b}{2\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t-t_a)f(t)\mathrm{d}t\\
&+\frac{1}{2m\omega\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t_b-t')f(t')\mathrm{d}t'\int_{t_a}^{t_b}\sin\omega(t_a-t)f(t)\mathrm{d}t\\
&+\frac{1}{2m\omega}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t-t')f(t')f(t)\mathrm{d}t'\mathrm{d}t. \qquad (4.27)
\end{aligned}
$$

Sum over the $4^{th}, 5^{th}$and 6^{th} terms of equation (4.27) into a single term

$$
\begin{aligned}
&\frac{x_b}{2}\int_{t_a}^{t_b}\cos\omega(t_b-t')f(t')\mathrm{d}t'-\frac{x_b}{2}\frac{\cos\omega(t_b-t_a)}{\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t_b-t')f(t')\mathrm{d}t'\\
&+\frac{x_b}{2\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t-t_a)f(t)\mathrm{d}t\\
&\quad=\frac{x_b}{2\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t_b-t_a)\cos\omega(t_b-t')-\sin\omega(t_b-t')\cos\omega(t_b-t_a)f(t')\mathrm{d}t'\\
&\quad\quad+\frac{x_b}{2\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t-t_a)f(t)\mathrm{d}t\\
&\quad=\frac{x_b}{\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\sin\omega(t'-t_a)f(t')\mathrm{d}t'. \qquad (4.28)
\end{aligned}
$$

Consider the last two terms

$$\frac{1}{2m\omega\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\int_{t_a}^{t_b}\sin\omega(t_b-t')\sin\omega(t_a-t)f(t')f(t)\mathrm{d}t'\mathrm{d}t$$
$$+\frac{1}{2m\omega}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t-t')f(t')f(t)\mathrm{d}t'\mathrm{d}t$$
$$=\frac{1}{2m\omega\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t_b-t')\sin\omega(t_a-t)f(t')f(t)\mathrm{d}t'\mathrm{d}t$$
$$+\frac{1}{2m\omega\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\int_{t}^{t_b}\sin\omega(t_b-t')\sin\omega(t_a-t)f(t')f(t)\mathrm{d}t'\mathrm{d}t$$
$$+\frac{1}{2m\omega}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t-t')f(t')f(t)\mathrm{d}t'\mathrm{d}t. \tag{4.29}$$

The double integration of the second term of equation (4.29) can be rewritten by changing the order of integration of the variables t and t' and then interchanging the identities of the variables $t \leftrightarrow t'$, as

$$\int_{t_a}^{t_b}\int_{t}^{t_b}\sin\omega(t_b-t')\sin\omega(t_a-t)f(t')f(t)\mathrm{d}t'\mathrm{d}t$$
$$=\int_{t_a}^{t_b}\int_{t_a}^{t'}\sin\omega(t_b-t')\sin\omega(t_a-t)f(t')f(t)\mathrm{d}t\mathrm{d}t'$$
$$=\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t_b-t)\sin\omega(t_a-t')f(t)f(t')\mathrm{d}t'\mathrm{d}t.$$

Then equation (4.29) becomes

$$\frac{1}{2\mathrm{m}\omega\sin\omega(t_b-t_a)}\int_{t_a}^{t_b}\int_{t_a}^{t_b}\sin\omega(t_b-t')\sin\omega(t_a-t)f(t')f(t)\mathrm{d}t'\mathrm{d}t$$
$$+\frac{1}{2m\omega}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t-t')f(t')f(t)\mathrm{d}t'\mathrm{d}t$$
$$=\frac{1}{2m\omega\sin\omega(t_b-t_a)}\Bigg\{\int_{t_a}^{t_b}\int_{t_a}^{t}\Bigg[\sin\omega(t_b-t')\sin\omega(t_a-t)$$
$$+\sin\omega(t_b-t)\sin\omega(t_a-t')+\sin\omega(t-t')\sin\omega(t_b-t_a)\Bigg]f(t')f(t)\mathrm{d}t'\mathrm{d}t\Bigg\}. \tag{4.30}$$

We expand the three terms of the integrand using formula

$$\sin A\sin B=\frac{1}{2}\Big[\cos(A-B)-\cos(A+B)\Big].$$

$$\Big[\sin\omega(t_b-t')\sin\omega(t_a-t)+\sin\omega(t_b-t)\sin\omega(t_a-t')+\sin\omega(t-t')\sin\omega(t_b-t_a)\Big]$$
$$=\frac{1}{2}\Big[\cos\omega(t_b-t'-t_a+t)-\cos\omega(t_b-t'+t_a-t)$$
$$+\cos\omega(t_b-t-t_a+t')-\cos\omega(t_b-t+t_a-t')$$
$$+\cos\omega(t-t'-t_b+t_a)-\cos\omega(t-t'+t_b-t_a)\Big]$$
$$=\cos\omega(t-t'-t_b+t_a)-\cos\omega(t_b+t_a-t'-t).$$

From

$$\cos A-\cos B=2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(B-A)$$

therefore

$$\cos\omega(t-t'-t_b+t_a)-\cos\omega(t_b+t_a-t'-t)=-2\sin\omega(t'-t_a)\sin\omega(t_b-t).$$

Substituting this result into equation (4.30), we get

$$\frac{1}{2m\omega\sin\omega(t_b-t_a)}\Big\{\int_{t_a}^{t_b}\int_{t_a}^{t}\Big[\sin\omega(t_b-t')\sin\omega(t_a-t)$$
$$+\sin\omega(t_b-t)\sin\omega(t_a-t')+\sin\omega(t-t')\sin\omega(t_b-t_a)\Big]f(t')f(t)\mathrm{d}t'\mathrm{d}t\Big\}$$
$$=-\frac{1}{m\omega\sin\omega(t_b-t_a)}\Big\{\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\Big\}. \tag{4.31}$$

Finally, the classical action of the forced harmonic oscillator is

$$S_{\rm cl} = \frac{m\omega}{2\sin\omega(t_b-t_a)}\Big[\Big(\cos\omega(t_b-t_a)\Big)\Big(x_b^2+x_a^2\Big)-2x_bx_a$$
$$+\frac{2x_a}{m\omega}\int_{t_a}^{t_b}\sin\omega(t_b-t)f(t)\mathrm{d}t$$
$$+\frac{2x_b}{m\omega}\int_{t_a}^{t_b}\sin\omega(t-t_a)f(t)\mathrm{d}t$$
$$-\frac{2}{m^2\omega^2}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\Big]. \tag{4.32}$$

4.2 The forced harmonic oscillator prefactor and propagator

From the relation

$$K(x_b,t_b;x_a,t_a)=\int_{-\infty}^{\infty}K(x_b,t_b;x_\mathrm{c},t_\mathrm{c})K(x_\mathrm{c},t_\mathrm{c};x_a,t_a)\mathrm{d}x_\mathrm{c}, \tag{4.33}$$

and the classical action calculated in the last section, we have

$$\begin{aligned}
&F(t_b,t_a)\mathrm{exp}\Bigg\{\frac{m\omega i}{2\hbar\sin\omega(t_b-t_a)}\Bigg[\cos\omega(t_b-t_a)\Big(x_b^2+x_a^2\Big)-2x_bx_a\\
&\qquad+\frac{2x_a}{m\omega}\int_{t_a}^{t_b}\sin\omega(t_b-t)f(t)\mathrm{d}t\\
&\qquad+\frac{2x_b}{m\omega}\int_{t_a}^{t_b}\sin\omega(t-t_a)f(t)\mathrm{d}t\\
&\qquad-\frac{2}{m^2\omega^2}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\Bigg]\Bigg\}\\
&=\int_{-\infty}^{\infty}F(t_b,t_\mathrm{c})\mathrm{exp}\Bigg\{\frac{m\omega i}{2\hbar\sin\omega(t_b-t_\mathrm{c})}\Bigg[\cos\omega(t_b-t_\mathrm{c})\Big(x_b^2+x_\mathrm{c}^2\Big)-2x_bx_\mathrm{c}\\
&\qquad+\frac{2x_\mathrm{c}}{m\omega}\int_{t_\mathrm{c}}^{t_b}\sin\omega(t_b-t)f(t)\mathrm{d}t\\
&\qquad+\frac{2x_b}{m\omega}\int_{t_\mathrm{c}}^{t_b}\sin\omega(t-t_\mathrm{c})f(t)\mathrm{d}t\\
&\qquad-\frac{2}{m^2\omega^2}\int_{t_\mathrm{c}}^{t_b}\int_{t_\mathrm{c}}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_\mathrm{c})f(t')f(t)\mathrm{d}t'\mathrm{d}t\Bigg]\Bigg\}\\
&\cdot F(t_\mathrm{c},t_a)\mathrm{exp}\Bigg\{\frac{m\omega i}{2\hbar\sin\omega(t_\mathrm{c}-t_a)}\Bigg[\cos\omega(t_\mathrm{c}-t_a)\Big(x_\mathrm{c}^2+x_a^2\Big)-2x_\mathrm{c}x_a\\
&\qquad+\frac{2x_a}{m\omega}\int_{t_a}^{t_\mathrm{c}}\sin\omega(t_\mathrm{c}-t)f(t)\mathrm{d}t\\
&\qquad+\frac{2x_\mathrm{c}}{m\omega}\int_{t_a}^{t_\mathrm{c}}\sin\omega(t-t_a)f(t)\mathrm{d}t\\
&\qquad-\frac{2}{m^2\omega^2}\int_{t_a}^{t_\mathrm{c}}\int_{t_a}^{t}\sin\omega(t_\mathrm{c}-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\Bigg]\Bigg\}\mathrm{d}x_\mathrm{c}
\end{aligned}$$

$$
\begin{aligned}
= F(t_b, t_{\mathrm{c}})F(t_{\mathrm{c}}, t_a)\exp\Bigg\{ \frac{m\omega i}{2\hbar\sin\omega(t_b - t_{\mathrm{c}})}\Bigg[& \cos\omega(t_b - t_{\mathrm{c}})x_b^2 \\
& + \frac{2x_b}{m\omega}\int_{t_{\mathrm{c}}}^{t_b} \sin\omega(t - t_{\mathrm{c}})f(t)\mathrm{d}t \\
& - \frac{2}{m^2\omega^2}\int_{t_{\mathrm{c}}}^{t_b}\int_{t_{\mathrm{c}}}^{t} \sin\omega(t_b - t)\sin\omega(t' - t_{\mathrm{c}})f(t')f(t)\mathrm{d}t'\mathrm{d}t\Bigg]\Bigg\} \\
\cdot\exp\Bigg\{ \frac{m\omega i}{2\hbar\sin\omega(t_{\mathrm{c}} - t_a)}\Bigg[& \cos\omega(t_{\mathrm{c}} - t_a)x_a^2 \\
& + \frac{2x_a}{m\omega}\int_{t_a}^{t_{\mathrm{c}}} \sin\omega(t_{\mathrm{c}} - t)f(t)\mathrm{d}t \\
& - \frac{2}{m^2\omega^2}\int_{t_a}^{t_{\mathrm{c}}}\int_{t_a}^{t} \sin\omega(t_{\mathrm{c}} - t)\sin\omega(t' - t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\Bigg]\Bigg\} \\
\cdot\int_{-\infty}^{\infty}\exp\Bigg\{ \frac{m\omega i}{2\hbar\sin\omega(t_b - t_{\mathrm{c}})}\Bigg[& \cos\omega(t_b - t_{\mathrm{c}})x_{\mathrm{c}}^2 - 2x_b x_{\mathrm{c}} \\
& + \frac{2x_{\mathrm{c}}}{m\omega}\int_{t_{\mathrm{c}}}^{t_b} \sin\omega(t_b - t)f(t)\mathrm{d}t\Bigg]\Bigg\} \\
\cdot\exp\Bigg\{ \frac{m\omega i}{2\hbar\sin\omega(t_{\mathrm{c}} - t_a)}\Bigg[& \cos\omega(t_{\mathrm{c}} - t_a)x_{\mathrm{c}}^2 - 2x_{\mathrm{c}} x_a \\
& + \frac{2x_{\mathrm{c}}}{m\omega}\int_{t_a}^{t_{\mathrm{c}}} \sin\omega(t - t_a)f(t)\mathrm{d}t\Bigg]\Bigg\}\mathrm{d}x_{\mathrm{c}}. \qquad (4.34)
\end{aligned}
$$

Since the prefactor $F(t_b,t_a)$ is independent of x_a and x_b so we can set $x_a = x_b = 0$ without altering the value of $F(t_b,t_a)$. Equation (4.34) then becomes

$$
\begin{aligned}
F(t_b,t_a)\exp&\left\{\frac{m\omega i}{2\hbar\sin\omega T}\left[-\frac{2}{m^2\omega^2}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\right]\right\}\\
&= F(t_b,t_c)F(t_c,t_a)\\
&\quad\exp\left\{\frac{m\omega i}{2\hbar\sin\omega T'}\left[-\frac{2}{m^2\omega^2}\int_{t_c}^{t_b}\int_{t_c}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_c)f(t')f(t)\mathrm{d}t'\mathrm{d}t\right]\right\}\\
&\quad\exp\left\{\frac{m\omega i}{2\hbar\sin\omega T''}\left[-\frac{2}{m^2\omega^2}\int_{t_a}^{t_c}\int_{t_a}^{t}\sin\omega(t_c-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\right]\right\}\\
&\quad\int_{-\infty}^{\infty}\exp\left\{\frac{m\omega i}{2\hbar\sin\omega T'}\left[\cos\omega T' x_c^2+\frac{2x_c}{m\omega}\int_{t_c}^{t_b}\sin\omega(t_b-t)f(t)\mathrm{d}t\right]\right\}\\
&\qquad\exp\left\{\frac{m\omega i}{2\hbar\sin\omega T''}\left[\cos\omega T'' x_c^2+\frac{2x_c}{m\omega}\int_{t_a}^{t_c}\sin\omega(t-t_a)f(t)\mathrm{d}t\right]\right\}\mathrm{d}x_c,\\
&= F(t_b,t_c)F(t_c,t_a)\\
&\quad\exp\left\{\frac{m\omega i}{2\hbar\sin\omega T'}\left[-\frac{2}{m^2\omega^2}\int_{t_c}^{t_b}\int_{t_c}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_c)f(t')f(t)\mathrm{d}t'\mathrm{d}t\right]\right\}\\
&\quad\exp\left\{\frac{m\omega i}{2\hbar\sin\omega T''}\left[-\frac{2}{m^2\omega^2}\int_{t_a}^{t_c}\int_{t_a}^{t}\sin\omega(t_c-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\right]\right\}\\
&\quad\int_{-\infty}^{\infty}\exp\left\{\frac{m\omega i}{2\hbar}\left[\frac{\sin\omega T}{\sin\omega T'\sin\omega T''}\right]x_c^2\right.\\
&\quad\left.+x_c\frac{i}{\hbar}\left[\frac{1}{\sin\omega T'}\int_{t_c}^{t_b}\sin\omega(t_b-t)f(t)\mathrm{d}t+\frac{1}{\sin\omega T''}\int_{t_a}^{t_c}\sin\omega(t-t_a)f(t)\mathrm{d}t\right]\right\}\mathrm{d}x_c,
\end{aligned}
\tag{4.35}
$$

where $T = t_b - t_a$, $T' = t_b - t_c$, and $T'' = t_c - t_a$.
The integral of equation (4.35) is in the form of Gaussian integral.
From

$$\int_{-\infty}^{\infty}\exp\left\{-ax^2+bx\right\}\mathrm{d}x = \sqrt{\frac{\pi}{a}}\exp\left\{\frac{b^2}{4a}\right\},$$

when

$$
\begin{aligned}
a &= \frac{m\omega\sin\omega T}{2i\hbar\sin\omega T'\sin\omega T''}\\
b &= \frac{i}{\hbar}\left[\frac{1}{\sin\omega T'}\int_{t_c}^{t_b}\sin\omega(t_b-t)f(t)\mathrm{d}t+\frac{1}{\sin\omega T''}\int_{t_a}^{t_c}\sin\omega(t-t_a)f(t)\mathrm{d}t\right].
\end{aligned}
$$

We get

$$F(t_b,t_a)\exp\Bigg\{-\frac{i}{m\omega\hbar\sin\omega\mathrm{T}}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(\mathrm{t}_b-\mathrm{t})\sin\omega(\mathrm{t}'-\mathrm{t}_a)\mathrm{f}(\mathrm{t}')\mathrm{f}(\mathrm{t})\mathrm{d}\mathrm{t}'\mathrm{d}\mathrm{t}\Bigg\}$$

$$= F(t_b,t_\mathrm{c})F(t_\mathrm{c},t_a)\exp\Bigg\{-\frac{i}{m\omega\hbar}\Bigg[\frac{1}{\sin\omega T'}\int_{t_\mathrm{c}}^{t_b}\int_{t_\mathrm{c}}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_\mathrm{c})f(t')f(t)\mathrm{d}t'\mathrm{d}t$$

$$+\frac{1}{\sin\omega T''}\int_{t_a}^{t_\mathrm{c}}\int_{t_a}^{t}\sin\omega(t_\mathrm{c}-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\Bigg]\Bigg\}$$

$$\sqrt{\frac{2\pi i\hbar\sin\omega T'\sin\omega T''}{m\omega\sin\omega T}}\exp\Bigg\{\Bigg[-\frac{i\sin\omega T'\sin\omega T''}{2m\omega\hbar\sin\omega T}\Bigg]$$

$$\Bigg[\frac{1}{\sin^2\omega T'}\int_{t_\mathrm{c}}^{t_b}\int_{t_\mathrm{c}}^{t_b}\sin\omega(t_b-t)\sin\omega(t_b-t')f(t)f(t')\mathrm{d}t'\mathrm{d}t$$

$$+\frac{2}{\sin\omega T'\sin\omega T''}\int_{t_\mathrm{c}}^{t_b}\sin\omega(t_b-t)f(t)\mathrm{d}t\int_{t_a}^{t_\mathrm{c}}\sin\omega(t-t_a)f(t)\mathrm{d}t$$

$$+\frac{1}{\sin^2\omega T''}\int_{t_a}^{t_\mathrm{c}}\int_{t_a}^{t_\mathrm{c}}\sin\omega(t-t_a)\sin\omega(t'-t_a)f(t)f(t')\mathrm{d}t'\mathrm{d}t\Bigg]\Bigg\}$$

$$\begin{aligned}\frac{F(t_b,t_a)}{F(t_b,t_\mathrm{c})F(t_\mathrm{c},t_a)} &= \sqrt{\frac{2\pi i\hbar\sin\omega T'}{m\omega}\frac{m\omega}{2\pi i\hbar\sin\omega T}\frac{2\pi i\hbar\sin\omega T''}{m\omega}}\\
&\exp\Bigg\{-\frac{i}{m\omega\hbar\sin\omega T'}\int_{t_\mathrm{c}}^{t_b}\int_{t_\mathrm{c}}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_\mathrm{c})f(t')f(t)\mathrm{d}t'\mathrm{d}t\\
&-\frac{i}{m\omega\hbar\sin\omega T''}\int_{t_a}^{t_\mathrm{c}}\int_{t_a}^{t}\sin\omega(t_\mathrm{c}-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\\
&+\frac{i}{m\omega\hbar\sin\omega T}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\\
&-\frac{i\sin\omega T''}{2m\omega\hbar\sin\omega T\sin\omega T'}\int_{t_\mathrm{c}}^{t_b}\int_{t_\mathrm{c}}^{t_b}\sin\omega(t_b-t)\sin\omega(t_b-t')f(t')f(t)\mathrm{d}t'\mathrm{d}t\\
&-\frac{i}{m\omega\hbar\sin\omega T}\int_{t_\mathrm{c}}^{t_b}\sin\omega(t_b-t)f(t)\mathrm{d}t\int_{t_a}^{t_\mathrm{c}}\sin\omega(t-t_a)f(t)\mathrm{d}t\\
&-\frac{i\sin\omega T'}{2m\omega\hbar\sin\omega T\sin\omega T''}\int_{t_a}^{t_\mathrm{c}}\int_{t_a}^{t_\mathrm{c}}\sin\omega(t-t_a)\sin\omega(t'-t_a)f(t)f(t')\mathrm{d}t'\mathrm{d}t\Bigg\}.\end{aligned} \tag{4.36}$$

Combining the second and the sixth terms together, then interchanging the order of integration of $\int_{t_a}^{t_c}\int_{t}^{t_c}\mathrm{d}t'\mathrm{d}t$ to $\int_{t_a}^{t_c}\int_{t_a}^{t'}\mathrm{d}t\mathrm{d}t'$ and changing $t' \leftrightarrow t$ back again, we have

$$
\begin{aligned}
&-\frac{i}{m\omega\hbar\sin\omega T''}\int_{t_a}^{t_c}\int_{t_a}^{t}\sin\omega(t_c-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\\
&-\frac{i\sin\omega T'}{2m\omega\hbar\sin\omega T\sin\omega T''}\left[\left(\int_{t_a}^{t_c}\int_{t_a}^{t}+\int_{t_a}^{t_c}\int_{t}^{t_c}\right)\left(\sin\omega(t-t_a)\sin\omega(t'-t_a)f(t')f(t)\right)\mathrm{d}t'\mathrm{d}t\right]\\
&\qquad=-\frac{i}{2m\omega\hbar\sin\omega T\sin\omega T''}\int_{t_a}^{t_c}\int_{t_a}^{t}\Big\{2\sin\omega T\sin\omega(t_c-t)\\
&\qquad\quad+\sin\omega T'\sin\omega(t-t_a)+\sin\omega T'\sin\omega(t-t_a)\Big\}\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\\
&\qquad=-\frac{i}{m\omega\hbar\sin\omega T}\int_{t_a}^{t_c}\int_{t_a}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t.
\end{aligned}
\tag{4.37}
$$

In a similar way the first and the fourth terms can be combined to give

$$
\begin{aligned}
&-\frac{i}{m\omega\hbar\sin\omega T'}\int_{t_c}^{t_b}\int_{t_c}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_c)f(t')f(t)\mathrm{d}t'\mathrm{d}t\\
&-\frac{i\sin\omega T''}{2m\omega\hbar\sin\omega T\sin\omega T'}\int_{t_c}^{t_b}\int_{t_c}^{t}\sin\omega(t_b-t)\sin\omega(t_b-t')f(t')f(t)\mathrm{d}t'\mathrm{d}t\\
&-\frac{i\sin\omega T''}{2m\omega\hbar\sin\omega T\sin\omega T'}\int_{t_c}^{t_b}\int_{t}^{t_b}\sin\omega(t_b-t)\sin\omega(t_b-t')f(t')f(t)\mathrm{d}t'\mathrm{d}t\\
&\quad=-\frac{i}{2m\omega\hbar\sin\omega T\sin\omega T'}\int_{t_c}^{t_b}\int_{t_c}^{t}\Big\{2\sin\omega T\sin\omega(t'-t_c)+\sin\omega T''\sin\omega(t_b-t')\\
&\quad\quad+\sin\omega T''\sin\omega(t_b-t')\Big\}\sin\omega(t_b-t)f(t')f(t)\mathrm{d}t'\mathrm{d}t\\
&\quad=-\frac{i}{m\omega\hbar\sin\omega T}\int_{t_c}^{t_b}\int_{t_c}^{t}\sin\omega(t'-t_a)\sin\omega(t_b-t)f(t')f(t)\mathrm{d}t'\mathrm{d}t.
\end{aligned}
\tag{4.38}
$$

Then equation (4.36) becomes

$$\begin{aligned}
\frac{F(t_b,t_a)}{F(t_b,t_c)F(t_c,t_a)} &= \sqrt{\frac{2\pi i\hbar\sin\omega T'}{m\omega}\frac{m\omega}{2\pi i\hbar\sin\omega T}\frac{2\pi i\hbar\sin\omega T''}{m\omega}} \\
&\quad \exp\Bigg\{-\frac{i}{m\omega\hbar\sin\omega T}\Bigg[\int_{t_c}^{t_b}\int_{t_c}^{t}+\int_{t_a}^{t_c}\int_{t_a}^{t}-\int_{t_a}^{t_b}\int_{t_a}^{t}+\int_{t_c}^{t_b}\int_{t_a}^{t_c}\Bigg] \\
&\quad \sin\omega(t_b-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\Bigg\} \\
&= \sqrt{\frac{2\pi i\hbar\sin\omega T'}{m\omega}\frac{m\omega}{2\pi i\hbar\sin\omega T}\frac{2\pi i\hbar\sin\omega T''}{m\omega}} \\
&\quad \exp\Bigg\{-\frac{i}{m\omega\hbar\sin\omega T}\Bigg[\int_{t_c}^{t_b}\int_{t_a}^{t}+\int_{t_a}^{t_c}\int_{t_a}^{t}-\int_{t_a}^{t_b}\int_{t_a}^{t}\Bigg] \\
&\quad \sin\omega(t_b-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\Bigg\} \\
&= \sqrt{\frac{2\pi i\hbar\sin\omega T'}{m\omega}\frac{m\omega}{2\pi i\hbar\sin\omega T}\frac{2\pi i\hbar\sin\omega T''}{m\omega}} \\
&\quad \exp\Bigg\{-\frac{i}{m\omega\hbar\sin\omega T}\Bigg[\int_{t_a}^{t_b}\int_{t_a}^{t}-\int_{t_a}^{t_b}\int_{t_a}^{t}\Bigg] \\
&\quad \sin\omega(t_b-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\Bigg\} \\
\frac{F(t_b,t_a)}{F(t_b,t_c)F(t_c,t_a)} &= \sqrt{\frac{2\pi i\hbar\sin\omega T'}{m\omega}\frac{m\omega}{2\pi i\hbar\sin\omega T}\frac{2\pi i\hbar\sin\omega T''}{m\omega}}.
\end{aligned}$$

Therefore the prefactor is

$$F(t_b,t_a)=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}. \tag{4.39}$$

Finally, the forced harmonic oscillator propagator can be written in the form

$$\begin{aligned}
K(x_b,t_b;x_a,t_a) &= \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}\exp\Bigg\{\frac{m\omega i}{2\hbar\sin\omega T}\Bigg[\cos\omega T\Big(x_b^2+x_a^2\Big)-2x_bx_a \\
&\quad +\frac{2x_a}{m\omega}\int_{t_a}^{t_b}\sin\omega(t_b-t)f(t)\mathrm{d}t \\
&\quad +\frac{2x_b}{m\omega}\int_{t_a}^{t_b}\sin\omega(t-t_a)f(t)\mathrm{d}t \\
&\quad -\frac{2}{m^2\omega^2}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\Bigg]\Bigg\},
\end{aligned} \tag{4.40}$$

where $T=t_\mathrm{b}-t_\mathrm{a}$.

4.3 The constant forced harmonic oscillator propagator

When the external force is constant $f(t) = \lambda = \text{constant}$, equation (4.40) becomes

$$\begin{aligned} K(x_b,t_b;x_a,t_a) &= \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}\exp\Bigg\{\frac{m\omega i}{2\hbar\sin\omega T}\Bigg[\cos\omega T\Big(x_b^2+x_a^2\Big)-2x_bx_a \\ &\qquad +\frac{2\lambda x_a}{m\omega^2}\cos\omega(t_b-t)\Big|_{t_a}^{t_b} \\ &\qquad -\frac{2\lambda x_b}{m\omega^2}\cos\omega(t-t_a)\Big|_{t_a}^{t_b} \\ &\quad -\frac{2\lambda^2}{m^2\omega^2}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_a)\mathrm{d}t'\mathrm{d}t\Bigg]\Bigg\} \\ K(x_b,t_b;x_a,t_a) &= \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}\exp\Bigg\{\frac{m\omega i}{2\hbar\sin\omega T}\Bigg[\cos\omega T\Big(x_b^2+x_a^2\Big)-2x_bx_a \\ &\qquad +\frac{2\lambda x_a}{m\omega^2}\Big(1-\cos\omega(t_b-t_a)\Big) \\ &\qquad -\frac{2\lambda x_b}{m\omega^2}\Big(\cos\omega(t_b-t_a)-1\Big) \\ &\quad -\frac{2\lambda^2}{m^2\omega^2}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_a)\mathrm{d}t'\mathrm{d}t\Bigg]\Bigg\}. \end{aligned} \tag{4.41}$$

Consider the last term of equation (4.41)

$$\begin{aligned} &-\frac{2\lambda^2}{m^2\omega^2}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t_b-t)\sin\omega(t'-t_a)\mathrm{d}t'\mathrm{d}t \\ &\quad = -\frac{2\lambda^2}{m^2\omega^2}\int_{t_a}^{t_b}\sin\omega(t_b-t)\Bigg(\int_{t_a}^{t}\sin\omega(t'-t_a)\mathrm{d}t'\Bigg)\mathrm{d}t \\ &\quad = \frac{2\lambda^2}{m^2\omega^3}\int_{t_a}^{t_b}\sin\omega(t_b-t)\Bigg(\cos\omega(t'-t_a)\Big|_{t_a}^{t}\Bigg)\mathrm{d}t \\ &\quad = \frac{2\lambda^2}{m^2\omega^3}\int_{t_a}^{t_b}\sin\omega(t_b-t)\Big(\cos\omega(t-t_a)-1\Big)\mathrm{d}t \\ &\quad = \frac{2\lambda^2}{m^2\omega^3}\Bigg[\int_{t_a}^{t_b}\sin\omega(t_b-t)\cos\omega(t-t_a)\mathrm{d}t-\int_{t_a}^{t_b}\sin\omega(t_b-t)\mathrm{d}t\Bigg]. \end{aligned} \tag{4.42}$$

From

$$\sin A \cos B = \frac{1}{2}\Big[\sin(A-B) + \sin(A+B)\Big]$$

$$\sin\omega(t_b - t)\cos\omega(t - t_a) = \frac{1}{2}\Big[\sin\omega(t_b + t_a - 2t) + \sin\omega(t_b - t_a)\Big],$$

equation (4.42) can be rewritten as

$$\begin{aligned}
&-\frac{2\lambda^2}{m^2\omega^2}\int_{t_a}^{t_b}\int_{t_a}^{t}\sin\omega(t_b - t)\sin\omega(t' - t_a)\mathrm{d}t'\mathrm{d}t \\
&\quad= \frac{\lambda^2}{m^2\omega^3}\int_{t_a}^{t_b}\sin\omega(t_b + t_a - 2t)\mathrm{d}t + \frac{\lambda^2}{m^2\omega^3}\int_{t_a}^{t_b}\sin\omega(t_b - t_a)\mathrm{d}t \\
&\quad\quad -\frac{2\lambda^2}{m^2\omega^3}\int_{t_a}^{t_b}\sin\omega(t_b - t)\mathrm{d}t \\
&\quad= \frac{\lambda^2}{2m^2\omega^4}\cos\omega(t_b + t_a - 2t)\Big|_{t=t_a}^{t=t_b} + \frac{\lambda^2}{m^2\omega^3}\sin\omega(t_b - t_a)\int_{t_a}^{t_b}\mathrm{d}t \\
&\quad\quad -\frac{2\lambda^2}{m^2\omega^4}\cos\omega(t_b - t)\Big|_{t=t_a}^{t=t_b} \\
&\quad= \frac{\lambda^2}{2m^2\omega^4}\Big(\cos\omega(t_a - t_b) - \cos\omega(t_b - t_a)\Big) + \frac{\lambda^2}{m^2\omega^3}\Big(t_b - t_a\Big)\sin\omega(t_b - t_a) \\
&\quad\quad -\frac{2\lambda^2}{m^2\omega^4}\Big(1 - \cos\omega(t_b - t_a)\Big) \\
&\quad= \frac{\lambda^2}{m^2\omega^3}(t_b - t_a)\sin\omega(t_b - t_a) - \frac{2\lambda^2}{m^2\omega^4}\Big(1 - \cos\omega(t_b - t_a)\Big).
\end{aligned}$$

Equation (4.41) becomes

$$\begin{aligned}
K(x_b,t_b;x_a,t_a) &= \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}\exp\Bigg\{\frac{m\omega i}{2\hbar\sin\omega T}\Bigg[\cos\omega T\Big(x_b^2+x_a^2\Big)-2x_bx_a \\
&\qquad +\frac{2\lambda x_a}{m\omega^2}\Big(1-\cos\omega T\Big) \\
&\qquad +\frac{2\lambda x_b}{m\omega^2}\Big(1-\cos\omega T\Big) \\
&\qquad +\frac{\lambda^2}{m^2\omega^3}\Big(T\sin\omega T\Big) \\
&\qquad -\frac{2\lambda^2}{m^2\omega^4}\Big(1-\cos\omega T\Big)\Bigg]\Bigg\} \\
&= \sqrt{\frac{\mathrm{m}\omega}{2\pi i\hbar\sin\omega T}}\exp\Bigg\{\frac{m\omega i}{2\hbar\sin\omega T}\Bigg[\cos\omega T\Big(x_b^2+x_a^2\Big)-2x_bx_a\Bigg] \\
&\qquad +\frac{i}{\hbar\sin\omega T}\Bigg[\Big(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\Big)\Big(1-\cos\omega T\Big)\Bigg] \\
&\qquad +\frac{i}{2\hbar}\Bigg[\frac{\lambda^2}{m\omega^2}T\Bigg]\Bigg\}.
\end{aligned}$$

Therefore the constant forced harmonic oscillator propagator when $f(t)=\lambda$ is

$$\begin{aligned}
K(x_b,t_b;x_a,t_a) &= \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}\exp\Bigg\{-\frac{m\omega}{2\hbar}\Bigg[\Big(x_b^2+x_a^2\Big)\frac{\cos\omega T}{i\sin\omega T}-\frac{2x_bx_a}{i\sin\omega T}\Bigg] \\
&\qquad -\frac{1}{\hbar}\Bigg[\Big(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\Big)\Big(\frac{1}{i\sin\omega T}-\frac{\cos\omega T}{i\sin\omega T}\Big)\Bigg] \\
&\qquad +\frac{i}{2\hbar}\Bigg[\frac{\lambda^2}{m\omega^2}T\Bigg]\Bigg\}.
\end{aligned} \tag{4.43}$$

4.4 The ground-state wave function and energy of the constant forced harmonic oscillator

From

$$\begin{aligned}
i\sin\omega T &= \frac{1}{2}e^{i\omega T}\Big(1-e^{-2i\omega T}\Big) \\
\cos\omega T &= \frac{1}{2}e^{i\omega T}\Big(1+e^{-2i\omega T}\Big).
\end{aligned}$$

Substitute the relation above into equation (4.43), we obtain

$$\begin{aligned} K(x_b,t_b;x_a,t_a) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\Big(1-e^{-2i\omega T}\Big)^{-1/2}\exp\left(-\frac{i\omega T}{2}\right) \\ &\exp\Bigg\{-\frac{m\omega}{2\hbar}\left[\left(x_b^2+x_a^2\right)\frac{1+e^{-2i\omega T}}{1-e^{-2i\omega T}}-\frac{4x_bx_ae^{-i\omega T}}{1-e^{-2i\omega T}}\right] \\ &\qquad -\frac{1}{\hbar}\left[\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(\frac{2e^{-i\omega T}}{1-e^{-2i\omega T}}-\frac{1+e^{-2i\omega T}}{1-e^{-2i\omega T}}\right)\right] \\ &\qquad +\frac{i}{2\hbar}\left(\frac{\lambda^2}{m\omega^2}T\right)\Bigg\}. \end{aligned} \tag{4.44}$$

In case of Taylor's series expansion of any function $f(x)$ expansion about any center x_0

$$f(x)=\sum_{n=0}^{\infty}\frac{\mathrm{d}^nf(x_0)}{\mathrm{d}x^n}\frac{(x-x_0)^n}{n!}$$

consider $\big(1-e^{-2i\omega T}\big)^{-1/2}$ in equation (4.44), we set $x=e^{-2i\omega T}$.
Power series of $\big(1-e^{-2i\omega T}\big)^{-1/2}$ is as follows

$$\begin{aligned} f(x) &= (1-x)^{-1/2} & f(0)&=1 \\ f'(x) &= \frac{1}{2}(1-x)^{-3/2} & f'(0)&=\frac{1}{2} \\ f''(x) &= \frac{1\cdot3}{2^2}(1-x)^{-5/2} & f''(0)&=\frac{1\cdot3}{2^2} \\ f'''(x) &= \frac{1\cdot3\cdot5}{2^3}(1-x)^{-7/2} & f'''(0)&=\frac{1\cdot3\cdot5}{2^3} \\ f^{(4)}(x) &= \frac{1\cdot3\cdot5\cdot7}{2^4}(1-x)^{-9/2} & f^{(4)}(0)&=\frac{1\cdot3\cdot5\cdot7}{2^4} \end{aligned}$$

$$(1-x)^{-1/2}=1+\frac{1}{2}x+\frac{1\cdot3}{2^22!}x^2+\frac{1\cdot3\cdot5}{2^33!}x^3+\frac{1\cdot3\cdot5\cdot7}{2^44!}x^4+...$$

So we can write

$$(1-e^{-2i\omega T})^{-1/2}=1+\frac{1}{2}e^{-2i\omega T}+\frac{1\cdot3}{2^22!}e^{-4i\omega T}+\frac{1\cdot3\cdot5}{2^33!}e^{-6i\omega T}+\frac{1\cdot3\cdot5\cdot7}{2^44!}e^{-8i\omega T}+... \tag{4.45}$$

In the same manner, we expand $\Big(1-e^{-2i\omega T}\Big)^{-1}$ in equation (4.44) (by setting $x=e^{-2i\omega T}$) as follows

$$
\begin{aligned}
f(x) &= (1-x)^{-1} & f(0)&=1\\
f'(x) &= (1-x)^{-2} & f'(0)&=1\\
f''(x) &= 2(1-x)^{-3} & f''(0)&=2\\
f'''(x) &= 6(1-x)^{-4} & f'''(0)&=6\\
f^{(4)}(x) &= 24(1-x)^{-5} & f^{(4)}(0)&=24
\end{aligned}
$$

$$
\begin{aligned}
\Big(1-x\Big)^{-1} &= 1+x+\frac{2x^2}{2!}+\frac{6x^3}{3!}+\frac{24x^4}{4!}+...\\
&= 1+x+x^2+x^3+x^4+... \ .\\
\Big(1-e^{-2i\omega T}\Big)^{-1} &= 1+e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...
\end{aligned}
$$

Therefore

$$
\left(\frac{1+e^{-2i\omega T}}{1-e^{-2i\omega T}}\right) = 1+e^{-2i\omega T}\Big(1+e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\Big).
$$

Now we expand $\left(\frac{-4x_bx_ae^{-i\omega T}}{1-e^{-2i\omega T}}\right)$ in the form of Taylor's series as follows

$$
\left(\frac{-4x_bx_ae^{-i\omega T}}{1-e^{-2i\omega T}}\right) = -4x_bx_ae^{-i\omega T}\Big(1+e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\Big).
$$

Similarly

$$
\left(\frac{2e^{-i\omega T}}{1-e^{-2i\omega T}}\right) = 2e^{-i\omega T}\Big(1+e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\Big).
$$

Equation (4.44) becomes

$$\begin{aligned}
K(x_b,t_b;x_a,t_a) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left(-\frac{i\omega T}{2}\right)\\
&\left(1+\frac{1}{2}e^{-2i\omega T}+\frac{1\cdot 3}{2^2 2!}e^{-4i\omega T}+\frac{1\cdot 3\cdot 5}{2^3 3!}e^{-6i\omega T}+\frac{1\cdot 3\cdot 5\cdot 7}{2^4 4!}e^{-8i\omega T}+...\right)\\
&\exp\left\{-\frac{m\omega}{2\hbar}\left[x_b^2+x_a^2\left(1+e^{-2i\omega T}\right)\left(1+e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+...\right)\right.\right.\\
&\left.-\left(4x_b x_a e^{-i\omega T}\right)\left(1+e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]\\
&-\frac{1}{\hbar}\left[\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\right]\\
&\left[2e^{-i\omega T}\left(1+e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right.\\
&\left.-\left(1+e^{-2i\omega T}\right)\left(1+e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]\\
&\left.+\frac{i}{2\hbar}\left[\frac{\lambda^2 T}{m\omega^2}\right]\right\}\\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left(-\frac{i\omega T}{2}\right)\\
&\left(1+\frac{1}{2}e^{-2i\omega T}+\frac{1\cdot 3}{2^2 2!}e^{-4i\omega T}+\frac{1\cdot 3\cdot 5}{2^3 3!}e^{-6i\omega T}+\frac{1\cdot 3\cdot 5\cdot 7}{2^4 4!}e^{-8i\omega T}+...\right)\\
&\exp\left\{-\frac{m\omega}{2\hbar}\left\{x_b^2+x_a^2\left[\left(1+e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right.\right.\right.\\
&\left.+\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]\\
&\left.-4x_b x_a\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right\}\\
&-\frac{1}{\hbar}\left[\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\right]\\
&\left[\left(2e^{-i\omega T}+2e^{-3i\omega T}+2e^{-5i\omega T}+2e^{-7i\omega T}+2e^{-9i\omega T}+...\right)\right.\\
&-\left\{\left(1+e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right.\\
&\left.\left.\left.+\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right\}\right]+\frac{i}{2\hbar}\left(\frac{\lambda^2 T}{m\omega^2}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
K(x_b,t_b;x_a,t_a) \;\; = \;\; & \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left(-\frac{i\omega T}{2}\right) \\
& \left(1+\frac{1}{2}e^{-2i\omega T}+\frac{1\cdot 3}{2^2 2!}e^{-4i\omega T}+\frac{1\cdot 3\cdot 5}{2^3 3!}e^{-6i\omega T}+\frac{1\cdot 3\cdot 5\cdot 7}{2^4 4!}e^{-8i\omega T}+...\right) \\
& \exp\left[-\frac{m\omega}{2\hbar}\left(x_b^2+x_a^2\right)\right] \\
& \exp\left[-\frac{m\omega}{\hbar}\left(x_b^2+x_a^2\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right. \\
& \quad +\frac{2m\omega}{\hbar}x_b x_a\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right) \\
& \quad -\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right) \\
& \quad \left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right) \\
& \quad +\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right) \\
& \quad \left.\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right] \\
& \exp\left[\frac{1}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\right]\exp\left[\frac{i}{2\hbar}\left(\frac{\lambda^2 T}{m\omega^2}\right)\right]. \qquad (4.46)
\end{aligned}
$$

Consider $\exp\left[-\frac{m\omega}{\hbar}\left(x_b^2+x_a^2\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]$ in equation (4.46), we set $x=-\frac{m\omega}{\hbar}\left(x_b^2+x_a^2\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)$ using Taylor's series expansion about $x=0$,

$$
e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+...
$$

Then

$$
\begin{aligned}
& \exp\left[-\frac{m\omega}{\hbar}\left(x_b^2+x_a^2\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right] \\
& \quad = 1+\left[-\frac{m\omega}{\hbar}\left(x_b^2+x_a^2\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right] \\
& \qquad +\frac{1}{2!}\left[-\frac{m\omega}{\hbar}\left(x_b^2+x_a^2\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]^2 \\
& \qquad +\frac{1}{3!}\left[-\frac{m\omega}{\hbar}\left(x_b^2+x_a^2\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]^3 \\
& \qquad +... \;\; .
\end{aligned}
\qquad (4.47)
$$

Then we expand $\exp\left[\frac{2m\omega}{\hbar}x_b x_a\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]$ by the same method with equation (4.47) as

$$\begin{aligned}
&\exp\left[\frac{2m\omega}{\hbar}x_b x_a\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]\\
&\qquad=1+\left[\frac{2m\omega}{\hbar}x_b x_a\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]\\
&\qquad\quad+\frac{1}{2!}\left[\frac{2m\omega}{\hbar}x_b x_a\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]^2\\
&\qquad\quad+\frac{1}{3!}\left[\frac{2m\omega}{\hbar}x_b x_a\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]^3\\
&\qquad\quad+...
\end{aligned} \tag{4.48}$$

Consider $\exp\left[-\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]$, the result of the expansion is

$$\begin{aligned}
&\exp\left[-\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]\\
&\qquad=1+\left[-\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]\\
&\qquad\quad+\frac{1}{2!}\left[-\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]^2\\
&\qquad\quad+\frac{1}{3!}\left[-\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]^3\\
&\qquad\quad+...
\end{aligned} \tag{4.49}$$

Then we consider $\exp\left[\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]$, it yields

$$\begin{aligned}
&\exp\left[\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right] \\
&\qquad = 1+\left[\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right] \\
&\qquad\quad +\frac{1}{2!}\left[\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]^2 \\
&\qquad\quad +\frac{1}{3!}\left[\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]^3 \\
&\qquad\quad +...
\end{aligned} \tag{4.50}$$

Substitute the whole expansion results into equation (4.46), we obtain

$$
\begin{aligned}
K(x_b,t_b;x_a,t_a) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(1+\frac{1}{2}e^{-2i\omega T}+\frac{1\cdot 3}{2^2 2!}e^{-4i\omega T}+\frac{1\cdot 3\cdot 5}{2^3 3!}e^{-6i\omega T}+...\right) \\
&\exp\left[-\frac{i\omega T}{2}-\frac{m\omega}{2\hbar}\left(x_b^2+x_a^2\right)+\frac{1}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)+\frac{i}{2\hbar}\left(\frac{\lambda^2 T}{m\omega^2}\right)\right] \\
&\left\{1+\left[-\frac{m\omega}{\hbar}\left(x_b^2+x_a^2\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]\right. \\
&\quad+\frac{1}{2!}\left[-\frac{m\omega}{\hbar}\left(x_b^2+x_a^2\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]^2 \\
&\quad\left.+\frac{1}{3!}\left[-\frac{m\omega}{\hbar}\left(x_b^2+x_a^2\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]^3+...\right\} \\
&\left\{1+\left[\frac{2m\omega}{\hbar}x_b x_a\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]\right. \\
&\quad+\frac{1}{2!}\left[\frac{2m\omega}{\hbar}x_b x_a\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]^2 \\
&\quad\left.+\frac{1}{3!}\left[\frac{2m\omega}{\hbar}x_b x_a\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]^3+...\right\} \\
&\left\{1+\left[-\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]\right. \\
&\quad+\frac{1}{2!}\left[-\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]^2 \\
&\quad+\frac{1}{3!}\left[-\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-i\omega T}+e^{-3i\omega T}+e^{-5i\omega T}+e^{-7i\omega T}+...\right)\right]^3 \\
&\qquad\left.+...\right\} \\
&\left\{1+\left[\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]\right. \\
&\quad+\frac{1}{2!}\left[\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]^2 \\
&\quad+\frac{1}{3!}\left[\frac{2}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(e^{-2i\omega T}+e^{-4i\omega T}+e^{-6i\omega T}+e^{-8i\omega T}+...\right)\right]^3 \\
&\qquad\left.+...\right\}.
\end{aligned}
\tag{4.51}
$$

The propagator can be expanded in the form of wave functions and energies as

$$K(x_b, t_b; x_a, t_a) = \sum_{n=0}^{\infty} \phi_n(x_b)\phi_n^*(x_a)\exp\big(-iE_nT/\hbar\big),$$

where

$\phi_n(x)$ is the wave function.

E_n is the eigenenergy.

For the term with n = 0, that is $\phi_0(x_b)\phi_0^*(x_a)\exp\big(-iE_0T/\hbar\big)$.

Collect all the terms that involve the factor $\exp\big(iE_0T/\hbar\big)$, we have

$$\begin{aligned}
&\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\exp\left[-\frac{i\omega T}{2}-\frac{m\omega}{2\hbar}\left(x_b^2+x_a^2\right)+\frac{1}{\hbar}\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)+\frac{i}{2\hbar}\left(\frac{\lambda^2T}{m\omega^2}\right)\right]\\
&=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\left(\frac{m\omega}{2\hbar}x_b^2-\frac{\lambda x_b}{\hbar\omega}+\frac{\lambda^2}{2m\hbar\omega^3}\right)\right]\\
&\quad\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\left(\frac{m\omega}{2\hbar}x_a^2-\frac{\lambda x_a}{\hbar\omega}+\frac{\lambda^2}{2m\hbar\omega^3}\right)\right]\\
&\qquad\exp\left[\frac{i}{2\hbar}\left(\frac{\lambda^2T}{m\omega^2}\right)-\frac{i\omega T}{2}\right]\\
&=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{1}{2m\hbar\omega^3}\left(m^2\omega^4x_b^2-2\lambda m\omega^2x_b+\lambda^2\right)\right]\\
&\quad\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{1}{2m\hbar\omega^3}\left(m^2\omega^4x_a^2-2\lambda m\omega^2x_a+\lambda^2\right)\right]\\
&\qquad\exp\left[-\frac{i}{\hbar}\left(\frac{m\hbar\omega^3-\lambda^2}{2m\omega^2}\right)T\right]\\
&=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{1}{2m\hbar\omega^3}\left(m\omega^2x_b-\lambda\right)^2\right]\\
&\quad\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{1}{2m\hbar\omega^3}\left(m\omega^2x_a-\lambda\right)^2\right]\\
&\qquad\exp\left[-\frac{i}{\hbar}\left(\frac{\hbar\omega}{2}-\frac{\lambda^2}{2m\omega^2}\right)T\right]\\
&=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x_b-\frac{\lambda}{m\omega^2}\right)^2\right]\\
&\quad\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x_a-\frac{\lambda}{m\omega^2}\right)^2\right]\\
&\qquad\exp\left[-\frac{i}{\hbar}\left(\frac{\hbar\omega}{2}-\frac{\lambda^2}{2m\omega^2}\right)T\right].
\end{aligned}$$

Hence

$$\begin{aligned}\phi_0(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x-\frac{\lambda}{m\omega^2}\right)^2\right]\\ E_0 &= \frac{\hbar\omega}{2}-\frac{\lambda^2}{2m\omega^2}.\end{aligned}$$

4.5 Alternative method for calculating the ground-state wave function and energy of the constant forced harmonic oscillator

From

$$K(x_b,t_b;x_a,t_a)=\sum_{\mathrm{n}=0}^{\infty}\phi_{\mathrm{n}}(x_b)\phi_{\mathrm{n}}^*(x_a)e^{-iE_{\mathrm{n}}T/\hbar}$$

Let $T=-i\beta\hbar$ and substitute into the equation above, then

$$\begin{aligned}K(x_b,x_a;T) &= K(x_b,x_a;-i\beta\hbar)\\ &= \rho(x_b,x_a;\beta)\\ &= \sum_{\mathrm{n}=0}^{\infty}\phi_{\mathrm{n}}(x_b)\phi_{\mathrm{n}}^*(x_a)e^{-\beta E_{\mathrm{n}}}\\ &= \phi_0(x_b)\phi_0^*(x_a)e^{-\beta E_0}+\phi_1(x_b)\phi_1^*(x_a)e^{-\beta E_1}+\phi_2(x_b)\phi_2^*(x_a)e^{-\beta E_2}+...,\end{aligned}$$

where $E_0<E_1<E_2<...$

When $\beta\to\infty$, the factors $e^{-\beta E_0}, e^{-\beta E_1}, e^{-\beta E_2}, ...$ decay to zero. Since we are interested in the ground state and $e^{-\beta E_0}>e^{-\beta E_1}>e^{-\beta E_2}>...$, we keep only the first term of the expansion. So $\rho(x_b,x_a;\beta)$ can be approximated to be

$$\rho(x_b,x_a;\beta)\simeq\phi_0(x_b)\phi_0^*(x_a)e^{-\beta E_0}.$$

From equation (4.44), we let $T = -i\beta\hbar$, that is

$$\begin{aligned}\rho(x_b, x_a; \beta) \;=\; & \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{\beta\hbar\omega}{2}\right) \\ & \left(1 - e^{-2\beta\hbar\omega}\right)^{-1/2} \exp\Bigg\{-\frac{m\omega}{2\hbar}\left[\left(x_b^2 + x_a^2\right)\frac{1 + e^{-2\beta\hbar\omega}}{1 - e^{-2\beta\hbar\omega}} - \frac{4x_b x_a e^{-\beta\hbar\omega}}{1 - e^{-2\beta\hbar\omega}}\right] \\ & \qquad -\frac{1}{\hbar}\left[\left(\frac{\lambda x_a}{\omega} + \frac{\lambda x_b}{\omega} - \frac{\lambda^2}{m\omega^3}\right)\left(\frac{2e^{-\beta\hbar\omega}}{1 - e^{-2\beta\hbar\omega}} - \frac{1 + e^{-2\beta\hbar\omega}}{1 - e^{-2\beta\hbar\omega}}\right)\right] \\ & \qquad + \frac{\beta\lambda^2}{2m\omega^2}\Bigg\}. \qquad (4.52)\end{aligned}$$

Take $\beta \to \infty$ and keep the term $\exp\big(-\frac{\beta\hbar\omega}{2}\big)$, equation (4.52) becomes

$$\rho(x_b, x_a;\beta) \simeq \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left[-\frac{\beta\hbar\omega}{2} - \frac{m\omega}{2\hbar}\left(x_b^2 + x_a^2\right) + \frac{1}{\hbar}\left(\frac{\lambda x_a}{\omega} + \frac{\lambda x_b}{\omega} - \frac{\lambda^2}{m\omega^3}\right) + \frac{\beta\lambda^2}{2m\omega^2}\right]$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\left(\frac{m\omega}{2\hbar}x_b^2 - \frac{\lambda x_b}{\hbar\omega} + \frac{\lambda^2}{2m\hbar\omega^3}\right)\right]$$
$$\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\left(\frac{m\omega}{2\hbar}x_a^2 - \frac{\lambda x_a}{\hbar\omega} + \frac{\lambda^2}{2m\hbar\omega^3}\right)\right]$$
$$\exp\left[-\frac{\beta\hbar\omega}{2} + \frac{\beta\lambda^2}{2m\omega^2}\right]$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{1}{2m\hbar\omega^3}\left(m^2\omega^4 x_b^2 - 2\lambda m\omega^2 x_b + \lambda^2\right)\right]$$
$$\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{1}{2m\hbar\omega^3}\left(m^2\omega^4 x_a^2 - 2\lambda m\omega^2 x_a + \lambda^2\right)\right]$$
$$\exp\left[-\beta\left(\frac{\hbar\omega}{2} - \frac{\lambda^2}{2m\omega^2}\right)\right]$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{1}{2m\hbar\omega^3}\left(m\omega^2 x_b - \lambda\right)^2\right]$$
$$\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{1}{2m\hbar\omega^3}\left(m\omega^2 x_a - \lambda\right)^2\right]$$
$$\exp\left[-\beta\left(\frac{\hbar\omega}{2} - \frac{\lambda^2}{2m\omega^2}\right)\right]$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x_b - \frac{\lambda}{m\omega^2}\right)^2\right]$$
$$\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x_a - \frac{\lambda}{m\omega^2}\right)^2\right]$$
$$\exp\left[-\beta\left(\frac{\hbar\omega}{2} - \frac{\lambda^2}{2m\omega^2}\right)\right].$$

Hence

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x - \frac{\lambda}{m\omega^2}\right)^2\right]$$
$$E_0 = \frac{\hbar\omega}{2} - \frac{\lambda^2}{2m\omega^2}.$$

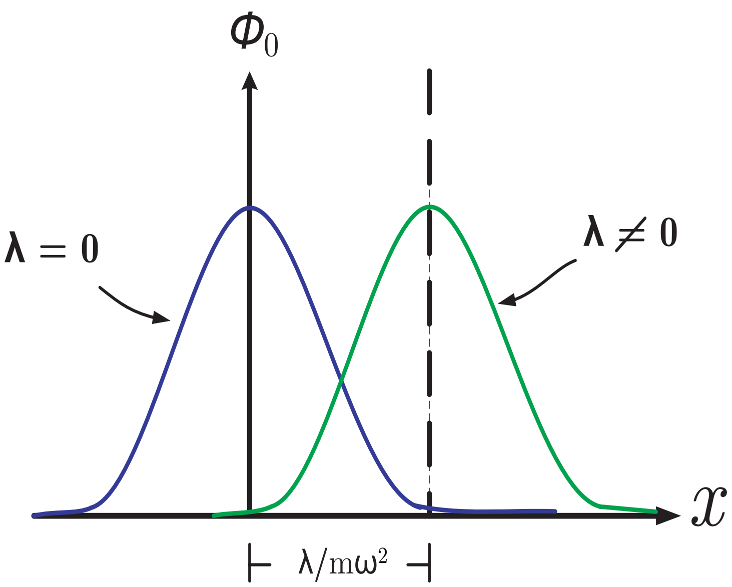

Figure 4.1: Position shift of the ground-state wave function when the external force $f(t) = \lambda =$ const. compared with no external force ($\lambda = 0$).

4.6 Derivation of the wave functions and energies of the constant forced harmonic oscillator from Schrödinger's equation

When

$$V = \frac{1}{2}m\omega^2 x^2 - \lambda x,$$

Schrödinger's equation is

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \left(\frac{1}{2}m\omega^2 x^2 - \lambda x\right)\phi = E\phi. \tag{4.53}$$

Consider $\left(\frac{1}{2}m\omega^2x^2 - \lambda x\right)$

$$\begin{aligned}
\frac{1}{2}m\omega^2x^2 - \lambda x &= \frac{1}{2}m\omega^2\left[x^2 - \frac{2x\lambda}{m\omega^2} + \left(\frac{\lambda}{m\omega^2}\right)^2 - \left(\frac{\lambda}{m\omega^2}\right)^2\right] \\
&= \frac{1}{2}m\omega^2\left[\left(x^2 - \frac{2x\lambda}{m\omega^2} + \frac{\lambda^2}{m^2\omega^4}\right) - \frac{\lambda^2}{m^2\omega^4}\right] \\
&= \frac{1}{2}m\omega^2\left[\left(x - \frac{\lambda}{m\omega^2}\right)^2 - \frac{\lambda^2}{m^2\omega^4}\right] \\
\frac{1}{2}m\omega^2x^2 - \lambda x &= \frac{1}{2}m\omega^2\left(x - \frac{\lambda}{m\omega^2}\right)^2 - \frac{\lambda^2}{2m\omega^2}. \qquad (4.54)
\end{aligned}$$

Substitute equation (4.54) into equation (4.53), that is

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \left[\frac{1}{2}m\omega^2\left(x - \frac{\lambda}{m\omega^2}\right)^2 - \frac{\lambda^2}{2m\omega^2}\right]\phi &= E\phi \\
-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + \frac{1}{2}m\omega^2\left(x - \frac{\lambda}{m\omega^2}\right)^2\phi &= \left(E + \frac{\lambda^2}{2m\omega^2}\right)\phi.
\end{aligned}$$

We set $y = x - \lambda/m\omega^2$, then

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2} + \frac{1}{2}m\omega^2y^2\phi = \left(E + \frac{\lambda^2}{2m\omega^2}\right)\phi. \qquad (4.55)$$

This is the Schrödinger's equation of the harmonic oscillator with energy $\left(E + \lambda^2/2m\omega^2\right)$ and wave functions are

$$\phi_n = (2^n n!)^{-1/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}H_n\left[\left(x - \frac{\lambda}{m\omega^2}\right)\left(\frac{m\omega}{\hbar}\right)^{1/2}\right]\exp\left[-\frac{m\omega}{2\hbar}\left(x - \frac{\lambda}{m\omega^2}\right)^2\right], \qquad (4.56)$$

where the function H_n are the Hermite polynomials

$$\begin{aligned}
H_0(y) &= 1 \\
H_1(y) &= 2y \\
H_2(y) &= 4y^2 - 2 \\
H_3(y) &= 8y^3 - 12y \\
&\cdot \\
&\cdot \\
&\cdot \\
H_n(y) &= (-1)^n\exp(y^2)\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left[\exp(-y^2)\right].
\end{aligned}$$

From the right-hand side of equation (4.55), we get

$$\begin{aligned} E + \frac{\lambda^2}{2m\omega^2} &= \hbar\omega\left(n + \frac{1}{2}\right) \\ E_n &= \hbar\omega\left(n + \frac{1}{2}\right) - \frac{\lambda^2}{2m\omega^2}, n = 0, 1, 2, ... \end{aligned}$$

For the ground state, we have

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega}{2\hbar}\left(x - \frac{\lambda}{m\omega^2}\right)^2\right],$$

and

$$E_0 = \frac{\hbar\omega}{2} - \frac{\lambda^2}{2m\omega^2},$$

as derived from the path integral method.

Chapter 5

Conclusions

In this work, we study the single-particle motion in the forced harmonic potential using Feynman path integral method. The Lagrangian for such a system is given by

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 + f(t)x,$$

where $f(t)$ is the external force exerting on the particle of mass m. As the Lagrangian has quadratic form, so the propagator of the system can be solved exactly as

$$K(x_b, t_b; x_a, t_a) = F(t_b, t_a)\exp\Big(\frac{i}{\hbar}S_{\rm cl}\Big).$$

From Green's function approach to the calculation, we obtain

$$\begin{aligned} S_{\rm cl} \;=\; & \frac{m\omega}{2\sin\omega T}\Bigg[\Big(\cos\omega T\Big)\Big(x_b^2 + x_a^2\Big) - 2x_b x_a \\ & \qquad + \frac{2x_a}{m\omega}\int_{t_a}^{t_b} \sin\omega(t_b - t)f(t)\mathrm{d}t \\ & \qquad + \frac{2x_b}{m\omega}\int_{t_a}^{t_b} \sin\omega(t - t_a)f(t)\mathrm{d}t \\ & - \frac{2}{m^2\omega^2}\int_{t_a}^{t_b}\int_{t_a}^{t} \sin\omega(t_b - t)\sin\omega(t' - t_a)f(t')f(t)\mathrm{d}t'\mathrm{d}t\Bigg], \end{aligned}$$

and

$$F(t_b, t_a) = \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}$$

where $T = t_\mathrm{b} - t_\mathrm{a}$. In case of the constant forced harmonic oscillator, the propagator is

$$\begin{aligned} K(x_b,t_b;x_a,t_a) &= \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}\exp\Bigg\{-\frac{m\omega}{2\hbar}\left[\left(x_b^2+x_a^2\right)\frac{\cos\omega T}{i\sin\omega T}-\frac{2x_bx_a}{i\sin\omega T}\right] \\ &\quad -\frac{1}{\hbar}\left[\left(\frac{\lambda x_a}{\omega}+\frac{\lambda x_b}{\omega}-\frac{\lambda^2}{m\omega^3}\right)\left(\frac{1}{i\sin\omega T}-\frac{\cos\omega T}{i\sin\omega T}\right)\right] \\ &\quad +\frac{i}{2\hbar}\left(\frac{\lambda^2}{m\omega^2}T\right)\Bigg\}. \end{aligned}$$

We expand the propagator for the ground-state energy eigenfunction and energy eigenvalue as follows

$$\begin{aligned} \phi_0(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{m\omega}{2\hbar}\left(x-\frac{\lambda}{m\omega^2}\right)^2\right] \\ E_0 &= \frac{\hbar\omega}{2}-\frac{\lambda^2}{2m\omega^2}. \end{aligned}$$

We conclude that the energy eigenfunction and the energy eigenvalue are the same as those obtained from Schrödinger's equation.

Appendix A

Classical motion of constant forced harmonic oscillator

From equation (4.22) when $f(t') = \lambda$, we can write

$$
\begin{aligned}
x_{\rm cl}(t) &= \frac{1}{\sin\omega(t_b - t_a)}\Big[x_a \sin\omega(t_b - t) + x_b \sin\omega(t - t_a)\Big] \\
&\quad + \frac{1}{m\omega}\left[\frac{\sin\omega(t_a - t)}{\sin\omega(t_b - t_a)}\int_{t_a}^{t_b} \sin\omega(t_b - t')\lambda \mathrm{d}t' \right. \\
&\qquad \left. + \int_{t_a}^{t} \sin\omega(t - t')\lambda \mathrm{d}t'\right] \\
x_{\rm cl}(t) &= \frac{1}{\sin\omega(t_b - t_a)}\Big[x_a \sin\omega(t_b - t) + x_b \sin\omega(t - t_a)\Big] \\
&\quad + \frac{\lambda}{m\omega}\left\{\frac{\sin\omega(t_a - t)}{\sin\omega(t_b - t_a)}\left[\frac{1}{\omega}\cos\omega(t_b - t')\Big|_{t_a}^{t_b}\right] + \frac{1}{\omega}\cos\omega(t - t')\Big|_{t_a}^{t}\right\} \\
&= \frac{1}{\sin\omega(t_b - t_a)}\Big[x_a \sin\omega(t_b - t) + x_b \sin\omega(t - t_a)\Big] \\
&\quad + \frac{\lambda}{m\omega^2}\left\{\frac{\sin\omega(t_a - t)}{\sin\omega(t_b - t_a)}\Big[1 - \cos\omega(t_b - t_a)\Big] + \Big[1 - \cos\omega(t - t_a)\Big]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\sin\omega(t_b-t_a)}\Big[x_a\sin\omega(t_b-t)+x_b\sin\omega(t-t_a)\Big] \\
&\quad +\frac{\lambda}{m\omega^2\sin\omega(t_b-t_a)}\Bigg\{\Big[\sin\omega(t_a-t)-\sin\omega(t_a-t)\cos\omega(t_b-t_a)\Big] \\
&\quad +\Big[\sin\omega(t_b-t_a)-\sin\omega(t_b-t_a)\cos\omega(t-t_a)\Big]\Bigg\} \\
&= \frac{1}{\sin\omega(t_b-t_a)}\Big[x_a\sin\omega(t_b-t)+x_b\sin\omega(t-t_a)\Big] \\
&\quad +\frac{\lambda}{m\omega^2\sin\omega(t_b-t_a)}\Bigg[\sin\omega(t_a-t)+\sin\omega(t_b-t_a)+\sin\omega(t-t_b)\Bigg] \\
&= \frac{1}{\sin\omega(t_b-t_a)}\Big[x_a\sin\omega(t_b-t)+x_b\sin\omega(t-t_a)\Big] \\
&\quad +\frac{\lambda}{m\omega^2}\frac{\sin\omega(t_a-t)}{\sin\omega(t_b-t_a)}+\frac{\lambda}{m\omega^2}\frac{\sin\omega(t-t_b)}{\sin\omega(t_b-t_a)}+\frac{\lambda}{m\omega^2} \\
&= \frac{\sin\omega(t_b-t)}{\sin\omega(t_b-t_a)}\left(x_a-\frac{\lambda}{m\omega^2}\right) \\
&\quad +\frac{\sin\omega(t-t_a)}{\sin\omega(t_b-t_a)}\left(x_b-\frac{\lambda}{m\omega^2}\right)+\frac{\lambda}{m\omega^2}. \qquad \text{(A.1)}
\end{aligned}
$$

We set $c_1 = \frac{1}{\sin\omega(t_b-t_a)}\Big(x_a - \frac{\lambda}{m\omega^2}\Big)$ and $c_2 = \frac{1}{\sin\omega(t_b-t_a)}\Big(x_b - \frac{\lambda}{m\omega^2}\Big)$, equation (A.1) becomes

$$
\begin{aligned}
x_{\rm cl}(t) &= c_1\sin\omega(t_b-t)+c_2\sin\omega(t-t_a)+\frac{\lambda}{m\omega^2} \\
&= c_1\Big(\sin\omega t_b\cos\omega t-\cos\omega t_a\sin\omega t\Big)+c_2\Big(\sin\omega t\cos\omega t_a-\cos\omega t\sin\omega t_a\Big)+\frac{\lambda}{m\omega^2} \\
&= (c_2-c_1)\sin\omega t\cos\omega t_a+\cos\omega t(c_1\sin\omega t_b-c_2\sin\omega t_a)+\frac{\lambda}{m\omega^2} \\
&= C\sin\omega t+C'\cos\omega t+\frac{\lambda}{m\omega^2}, \qquad \text{(A.2)}
\end{aligned}
$$

where $C=(c_2-c_1)\cos\omega t_a$ and $C'=(c_1\sin\omega t_b-c_2\sin\omega t_a)$.
We assume $C=A\cos\phi$, $C'=A\sin\phi$, then

$$
\begin{aligned}
C\sin\omega t+C'\cos\omega t+\frac{\lambda}{m\omega^2} &= A\sin\omega t\cos\phi+A\cos\omega t\sin\phi+\frac{\lambda}{m\omega^2} \\
x_{\rm cl}(t) &= A\sin(\omega t+\phi)+\frac{\lambda}{m\omega^2} \qquad \text{(A.3)}
\end{aligned}
$$

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